Plan

1. Envelope Theorem: Unconstrained case
2. Envelope Theorem: Constrained case
3. Lagrange multipliers

Next week: Differential equations
(revise integration) – Appendix "Differential equations"

– Bordered Hessians: not in the curriculum
(skip [WB] 9.6 – 9.8, 9.11b)

Review:

i) Second order condition: (SOC)

$(\underline{x}^*; \lambda^*)$ satisfies FOC+C / FOC+C+CSC

$h(x_1, \ldots, x_n) = \mathcal{L}(x_1, \ldots, x_n; \lambda_1^*, \ldots, \lambda_m^*)$ concave (convex)

$$\Downarrow$$

$\underline{x}^*$ is max (min)

ii) NDCQ:

a) Lagrange: $\left.\begin{array}{l} x^2+y^2+z^2=6 \\ x+y+z=3 \end{array}\right\}$ $J = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}$ NDCQ: $\mathrm{rk}\, J = 2$

adm. pts. where NDCQ fails: $\mathrm{rk}\, J < 2 \iff$ all 2-minors $= 0$

none

$2x-2y=0$, $2y-2z=0$

$2x-2z=0$

$\boxed{x=y=z}$

$x=y=z=1$

$(1)^2 \cdot 3 = 3 \neq 6$

b) Kuhn-Tucker case:

$x^2+y^2+z^2 \leq 6$

$x+y+z \geq 3$

$J = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}$

Adm. pts. where NDCQ fails:

i) $x^2+y^2+z^2=6$ NDCQ: $\operatorname{rk} J = 2$ as in Lagrange case

$x+y+z=3$ NDCQ fails: $\operatorname{rk} J < 2$

(binding c)

ii) $x^2+y^2+z^2=6$ NDCQ: $\operatorname{rk}\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = 1$

$x+y+z>3$ ok $x=y=z=0$ (not adm. $0+0+0 \neq 6$)

iii) $x^2+y^2+z^2<6$ NDCQ: $\operatorname{rk}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = 1$

$x+y+z=3$ ok rank is 1 for all x,y,z

iv) $x^2+y^2+z^2<6$ no condition ok

$x+y+z>3$

① Envelope Thm: Unconstrained case

Ex: max $f(x) = 1+2x-x^2$

$f' = 2-2x = 0 \quad x=1$ $\qquad f_{max} = 2$ at $x=1$

$f'' = -2 < 0 \quad f$ concave

$x^* = 1 \qquad f^* = 2$

max: $f(x) = 1 + ax - x^2$ where a is a parameter

Alt 1: max $f(x) = 1 + ax - x^2$

For each given value of a, we want to find the maximum:

$f' = a - 2x = 0 \quad x = a/2$

$f'' = -2 < 0 \quad f$ concave

$x^*(a) = a/2$

$\boxed{x^*(a) \qquad f^*(a)}$

$f^*(a) = 1 + a \cdot a/2 - (a/2)^2$

$= 1 + \frac{1}{2}a^2 - \frac{1}{4}a^2$

$= 1 + \frac{1}{4}a^2$

$\underbrace{x^*(2) = 1}_{a=2} \qquad \underbrace{f^*(2) = 2}_{a=2}$

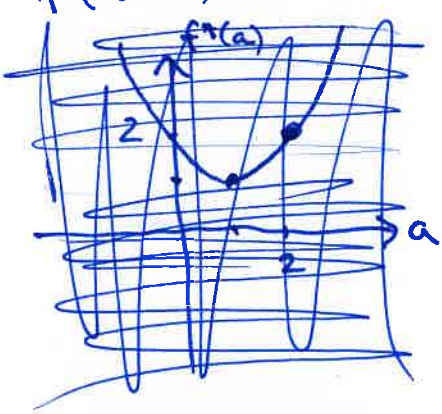
$f^*(a) =$ optimal value function $= f(x^*(a))$

Alt 2: Envelope Thm

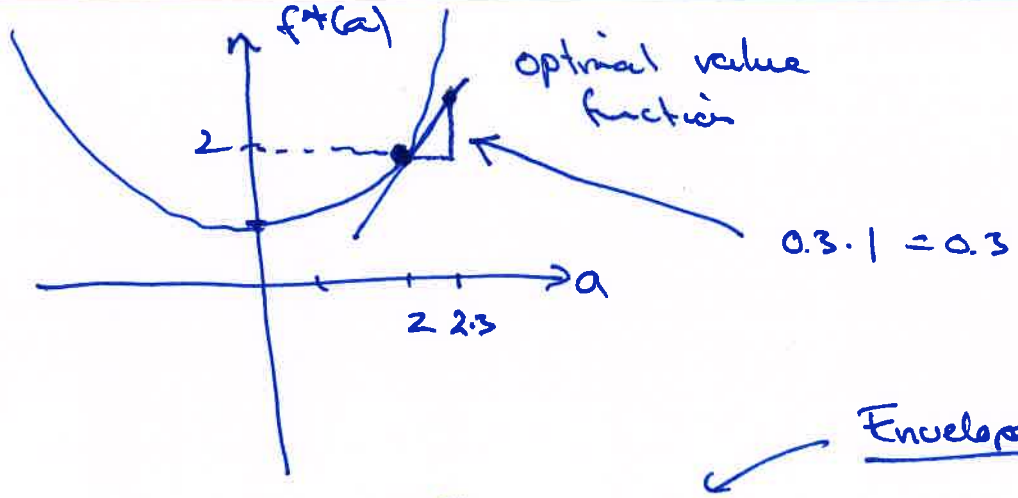
$f(x;a) = 1 + ax - x^2$

$f'_a = x \rightsquigarrow f'_a(x^*(a);a) = x^*(a)$

Env. Thm. $\Big\}$ $\dfrac{df^*(a)}{da} = f'_a(x^*(a);a) = f'_a(x^*(a))$



slope of tangent of $f^*(a)$ → $\dfrac{df^*(a)}{da} = x^*(a)$ ← we have to compute this



Envelope Thm

at $a = 2$:

$$\frac{df^*(a)}{da} = x^*(a) = 1$$

$x^* = x^*(2) = 1$

$f^* = f^*(2) = 2$

$$f^*(2.3) \simeq \underset{\substack{\uparrow \\ f^*(a) \\ \shortparallel \\ 2}}{f^*(2)} + \underset{\substack{\uparrow \\ \Delta a \\ \shortparallel \\ 2.3-2}}{0.3} \cdot \underset{\substack{\uparrow \\ \frac{df^*(a)}{da}}}{1} = \underline{\underline{2.3}}$$

Envelope Thm: Unconstrained case

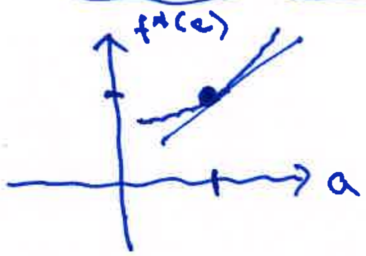
Assume that $\underline{x}^*(a)$ is the solution of the optimization problem

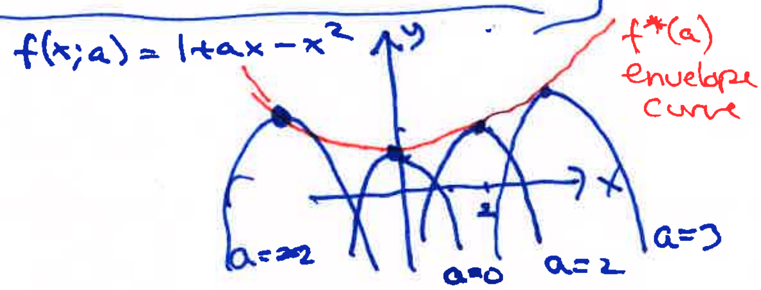
$$\max/\min \quad f(x_1, \dots, x_n; a)$$

Then

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(\underline{x}^*(a); a)$$

where $f^*(a) = f(x^*(a))$.





Ex: Kelly's Criterion

Game:
- win (prob. p) : double your money
- loose (prop $q = 1-p$) : loose your bet

Play repeatedly, bet a fixed share x of your capital each time.

$$f(x;p) = p \cdot \ln(1+x) + q \ln(1-x) \longleftarrow$$ expected exponential growth of capital

$$\boxed{\max f(x;p)}$$

$p = 0.60$: ($q = 0.40$)

$$\max f = 0.60 \ln(1+x) + 0.40 \ln(1-x)$$

$$f'_x = \frac{0.60}{1+x} + 0.40 \cdot \frac{1}{1-x} \cdot (-1) = \frac{0.60}{1+x} - \frac{0.40}{1-x}$$

$$= \frac{0.60 \cdot (1-x) - 0.40\,(1+x)}{(1-x^2)} = \frac{0.20 - x}{1-x^2} = 0$$

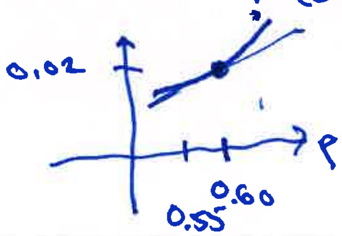
$$x = 0.20$$

$$f''_{xx} = -\frac{0.60}{(1+x)^2} - \frac{0.40}{(1-x)^2} < 0 \quad f \text{ concave}$$

Kelly bet. $\longrightarrow$ $x^* = 0.20$ $\quad f^* = f(0.20) \simeq 0.02$

What if $p = 0.55$: Envelope Thm

$p = 0.60$: $x^*(0.60) = 0.20$, $f^*(0.60) = 0.02$



$$f = p \cdot \ln(1+x) + (1-p) \cdot \ln(1-x)$$

$$f'_p = \ln(1+x) - \ln(1-x)$$

$p = 0.60$: $$\frac{df^*(p)}{dp} = f'_p(x^*(0.60))$$

$$= \ln(1+0.2) - \ln(1-0.2)$$

$$\simeq 0.41$$

Explanation: Kelly Criterion

X_0 : start capital

B_1 : bet (first game)

X_1 : capital after first game

win: $X_1 = X_0 + B_1$, $p(\text{win}) = p$

X_0

loose: $X_1 = X_0 - B_1$, $p(\text{loose}) = q = 1-p$

$$E(X_1) = p(X_0 + B_1) + (1-p)(X_0 - B_1)$$
$$= X_0 + pB_1 - (1-p)B_1$$
$$= X_0 + (p-q)B_1$$

assume $p - q > 0$ $(p > 0.5)$

$$E(X_1 - X_0) = (p-q)B_1 > 0$$

(expected return)

Play n games, each bet

$$B_i = x \cdot X_{i-1} \quad (0 \le x \le 1)$$

(ratio x of capital).

After 1 game:

W: $X_1 = X_0 + xX_0 = X_0(1+x)$

L: $X_1 = X_0 - xX_0 = X_0(1-x)$

After n games:

$$X_n = X_0 \cdot (1+x)^W \cdot (1-x)^L$$

$$\left.\begin{array}{l} W = \#\text{win} \\ L = \#\text{losses} \end{array}\right\} W + L = n$$

expected { exponential growth rate per game

$= f(x)$

$$E\left[\tfrac{1}{n}\ln(X_n/X_0)\right] = E\left[\tfrac{1}{n}\ln\left[(1+x)^W \cdot (1-x)^L\right]\right]$$
$$= E\left[\tfrac{1}{n}\left(W\ln(1+x) + L\cdot\ln(1-x)\right)\right]$$
$$= E\left[\tfrac{W}{n}\right]\ln(1+x) + E\left[\tfrac{L}{n}\right]\cdot\ln(1-x)$$
$$= p\,\ln(1+x) + q\cdot\ln(1-x) = f(x)$$

That means, $f(x) = g$

$\Downarrow$

"$X_n = X_0 e^{gn}$"

expected growth

Estimate: $f^*(0.55) \simeq f^*(0.60) + \Delta p \cdot \frac{df^*(p)}{dp}$

$= 0.02 + (-0.05) \cdot 0.41$

$= 0.02 - 0.020 \simeq 0.00$

(Exact value: $x^* = q - q = 0.55 - 0.45 = 0.10$ $\quad f^*(0.55) = f(0.10) \simeq 0.005$)

② Envelope Thm: Constrained cases

Ex: $\max\ (\min)\ f(x,y) = x + 3y$ w.h. $x^2 + y^2 = 10$ $\quad (x^2+y^2 \leq 10)$

$\mathcal{L} = x + 3y - \lambda(x^2+y^2-10)$

$\mathcal{L}'_x = 1 - \lambda \cdot 2x = 0 \qquad x = 1/2\lambda \quad (\lambda \neq 0)$

$\mathcal{L}'_y = 3 - \lambda \cdot 2y = 0 \qquad y = 3/2\lambda$

$x^2 + y^2 = 10 \qquad \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10 \qquad 1 \cdot (2\lambda)^2$

$1^2 + 3^2 = 10 \cdot (2\lambda)^2$

$10 = 10 \cdot (2\lambda)^2$

$(2\lambda)^2 = 1 \qquad 2\lambda = \pm 1 \qquad \lambda = \pm 1/2$

Cand. pts:

$(x,y;\lambda) = (1,3;1/2) \quad f = 10$

$(-1,-3;-1/2) \quad f = -10$

D: $x^2+y^2=10$ is bounded $\overset{EVT}{\Longrightarrow}$ there is a max $\underset{NDCQ}{\Longrightarrow}$ $f^* = 10$ at $(x^*,y^*) = (1,3)$, $\lambda^* = 1/2$

NDCQ:

$J = (2x \quad 2y)$

$\mathrm{rk}\, J = 1$

NDCQ fails: $x = y = 0$

$x^2+y^2 = 0^2+0^2 = 0 \neq 10$

NDCQ holds at all adm pts

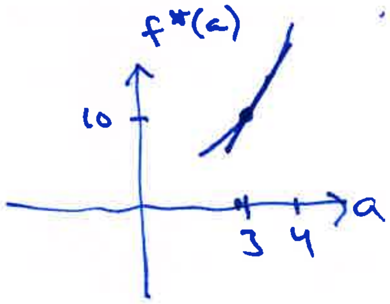
max $f(x,y) = x + 3y$ when $x^2 + y^2 = 10$

<u>What about:</u> i) max $x + 4y$ when $x^2+y^2 = 10$?
ii) max $x+3y$ when $x^2+y^2 = 11$?

i) max $f(x,y;a) = x + ay$ when $g(x,y) = x^2+y^2 = 10$

<u>$f^*(a)$:</u> optimal value function

$f^*(3) = 10$ $\quad \begin{cases} x^*(3) = 1 \\ y^*(3) = 3 \\ \lambda^*(3) = 1/2 \end{cases}$



$$\boxed{\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}\left(x^*(a), y^*(a); \lambda^*(a)\right)}$$

envelope thm

$L'_a = \Big(\underbrace{x + ay - \lambda(x^2+y^2-10)}_{L} \Big)'_a = y$

$f^*(4) \simeq \underset{10}{f^*(3)} + \underset{4-3}{\Delta a} \cdot \underset{3}{\frac{df^*(a)}{da}}$

$= 10 + 1 \cdot 3 = \underline{\underline{13}}$

$\frac{df^*(a)}{da} = y^*(a) \underset{a=3}{=} y^*(3) = \underline{3}$

ii) max $f(x,y;b) = x + 3y$ when $g(x,y;b) = x^2+y^2 = b$
$x^2+y^2-b = 0$

$$\boxed{\max\ f(x,y) = x + 3y \text{ when } x^2+y^2-b = 0}$$

$L = x + 3y - \lambda \cdot (x^2+y^2-b) = \dots + \lambda b$

<u>$b=10$:</u> $f^* = 10$ $\quad x^* = 1$ $\quad y^* = 3$ $\quad \lambda^* = 1/2$

$f^*(10) = 10$ $\quad x^*(10) = 1$ $\quad y^*(10) = 3$
$\underline{\lambda^*(10) = 1/2}$

<u>Envelope thm:</u> $L'_b = \lambda$

$\frac{df^*(b)}{db} = L'_b\left(x^*(b), y^*(b); \lambda^*(b)\right) = \lambda^*(b) = 1/2$

$f^*(11) \simeq \underset{10}{f^*(10)} + \underset{11-10}{\Delta b} \cdot \underset{1/2}{\frac{df^*(b)}{db}} = \underline{\underline{10.5}}$

<u>Envelope Thm:</u> Lagrange and Kuhn-Tucker problems

Assume that $(\underline{x}^*(a);\underline{\lambda}^*(a))$ is optimal point (max/min) in a Lagrange/Kuhn-Tucker problem <u>and</u> that it satisfies FOC+C / FOC+C+CSC.

Then:

$$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}(\underline{x}^*(a);\underline{\lambda}^*(a))$$

Where $f^*(a) = f(\underline{x}^*(a))$ and L contains a.

$L = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \ldots - \lambda_m(g_m(\underline{x}) - a_m)$

3 <u>Lagrange multipliers</u>

λ_i = marginal increase in max (min) value with respect to an increase in the constant in constraint # i.

<u>More exactly</u>: $\lambda_i = \dfrac{df^*(a)}{da}$ when a is the constant in constraint # i.