Selected problems:

(A) Key problems: 2.3c, 4.1ef, 4.3

(B) Workbook: 1.15, 2.18b, 2.21, 3.1, 3.2, 3.5, 3.12, 3.14, 4.3b, 4.5, 4.9, 4.10

(C) Midterm Exam: 10/2017 Question 1–5

Key problems:

2.3c) $A = \begin{pmatrix} \boxed{1} & a & b \\ a & b & 1 \end{pmatrix}$ Minors: max. minors = 2-minors

rk $A = 2$: at least one 2-minor is $\neq 0$

rk $A < 2$: all 2-minors are zero.

(1) $b = a^2$

(2) $1 = ab = a \cdot a^2$

$1 = a^3$

$a = 1$, $b = 1$

$$\begin{cases} M_{12,12} = \begin{vmatrix} 1 & a \\ a & b \end{vmatrix} = b - a^2 = 0 \\ M_{12,13} = \begin{vmatrix} 1 & b \\ a & 1 \end{vmatrix} = 1 - ab = 0 \\ M_{12,23} = \begin{vmatrix} a & b \\ b & 1 \end{vmatrix} = a - b^2 = 0 \end{cases}$$

$$\text{rk } A = \begin{cases} 2, & (a,b) \neq (1,1) \\ 1, & a=1, b=1 \text{ or } (a,b) = (1,1) \end{cases}$$

4.1 e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

42 e) Yes, diagonalizable

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

i) Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\cdot\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 2-\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$(2-\lambda)\cdot((2-\lambda)^2-1) - (2-\lambda-1) + (1-2+\lambda) = 0$$

$$(2-\lambda)\cdot(\lambda^2-4\lambda+3) + (\lambda-1) + (\lambda-1) = 0$$

$$(2-\lambda)\cdot(\lambda-1)(\lambda-3) + 2(\lambda-1) = 0$$

$$(\lambda-1)\cdot[(2-\lambda)(\lambda-3)+2] = 0$$

$$(\lambda-1)\cdot(-\lambda^2+5\lambda-4) = 0$$

$$-(\lambda-1)(\lambda^2-5\lambda+4) = 0 \longleftarrow$$

$$-(\lambda-1)^2(\lambda-4) = 0$$

$$-(\lambda^2-2\lambda+1)(\lambda-4) = 0$$

$$\boxed{-\lambda^3+6\lambda^2-9\lambda+4=0}$$

$\lambda=1 \quad \lambda=1 \quad \lambda=4$

ii) Eigenvectors:

$\lambda=1$: E_1

$$\begin{pmatrix} 1 & 1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{1} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A - 1\cdot\lambda)\cdot \underline{v} = \underline{0}$$

$x+y+z=0$, y,z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$B = \left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$ base for E_1

$\lambda=4$: E_4

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{matrix} -1 \\ \\ 2 \end{matrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ \cancel{0} & \cancel{3} & \cancel{-3} \end{pmatrix}$$

$x+y-2z=0$

$-3y+3z=0$

z free

$x=z$

$y=z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$B = \left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ for E_4

4.1 f) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 2$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$\lambda = 2$ (mult. 3)

Eigenvectors:

$\lambda = 2$: (E_2)

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

x free

$y + z = 0$ $\quad y = 0$

$z = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Base: $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

4.2 f) Not diagonalizable, $\dim E_2 = 1 < m = 3$

4.3 $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

A is diagonalizable since it is symmetric.

$$\begin{vmatrix} 1-\lambda & 0 & 0 & -1 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ -1 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \cdot \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \\ -1 & 0 & 0 \end{vmatrix}$$

$$(1-\lambda) \cdot (1-\lambda) \cdot [(1-\lambda)^2 - 1] + 1 \cdot (-1) [(1-\lambda)^2 - 1] = 0$$

$$((1-\lambda)^2 - 1) \cdot [\lambda^2 - 2\lambda \quad] = 0$$

$$(\lambda^2 - 2\lambda)(\lambda^2 - 2\lambda) = 0$$

$$\lambda \cdot (\lambda - 2)\, \lambda \cdot (\lambda - 2) = 0$$

$\Big\}$ $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 2$

BREAK

(3) Workbook

1.15.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right) \begin{array}{l} \\ \xleftarrow{-2} \\ \xleftarrow{-3} \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & -5 & p-3 & -5q \end{array}\right) \begin{array}{l} \\ \\ \xleftarrow{-1} \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & \boxed{p-3} & -5q \end{array}\right) \qquad \begin{aligned} x+y+z &= 2q \\ -5y &= 0 \\ (p-3)z &= -5q \end{aligned}$$

$p \neq 3$: one solution.

$p=3$: $\begin{cases} q \neq 0 & : \text{no solution} \\ q = 0 & : \text{inf. many solutions} \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -5q \end{array}\right)$$

$t=2$: $\begin{pmatrix} 5 & 5 & 6 \\ -1 & -1 & -6 \\ 1 & 1 & 6 \end{pmatrix}$, $\boxed{\begin{matrix} 5 & 6 \\ -1 & -6 \end{matrix}}$

2.18b

$$|A| = \left|\begin{pmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{pmatrix}\right| = 0 \iff rk < 3$$

$$\neq 0 \iff rk = 3$$

$$(t+3)\cdot\underbrace{\left((t-3)(t+4)+6\right)}_{\substack{t^2+t-6 \\ =(t-2)(t+3)}} \underbrace{-5\cdot\left(-(t+4)+6\right)+6\left(-1-(t-3)\right)}_{-t+2\;=\;-(t-2)} = 0$$

$$(t-2)\cdot\left((t+3)^2-1\right) = 0 \qquad t=2 \quad \text{or} \quad (t+3)^2 = 1$$

$$t+3 = \pm 1$$

$$t = \pm 1 - 3$$

$$t = -2, \quad t = -4$$

$$rk\,A = \begin{cases} 3, & t \neq 2, -2, -4 \\ 2, & t = 2, -2, -4 \end{cases}$$

$t=-2$: $\begin{pmatrix} 1 & 5 & 6 \\ -1 & -5 & -6 \\ 1 & 1 & 2 \end{pmatrix}$ $\qquad$ $t=-4$: $\begin{pmatrix} -1 & 5 & 6 \\ -1 & -7 & -6 \\ 1 & 1 & 0 \end{pmatrix}$

2.21:

$\underline{x}_1, \underline{x}_2$ solutions of $A\underline{x} = \underline{b}$

$\Downarrow$

$$\left.\begin{array}{l} A\underline{x}_1 = \underline{b} \\ A\underline{x}_2 = \underline{b} \end{array}\right\} \Longrightarrow$$

Straight line thr. $\underline{x}_1$ and $\underline{x}_2$

$\underline{x} = \lambda \underline{x}_1 + (1-\lambda)\underline{x}_2$ is solution of $A\underline{x} = \underline{b}$ for all λ:

$$A \cdot \underline{x} =$$

$$A \cdot (\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2) = A \cdot \lambda \underline{x}_1 + A(1-\lambda)\underline{x}_2$$

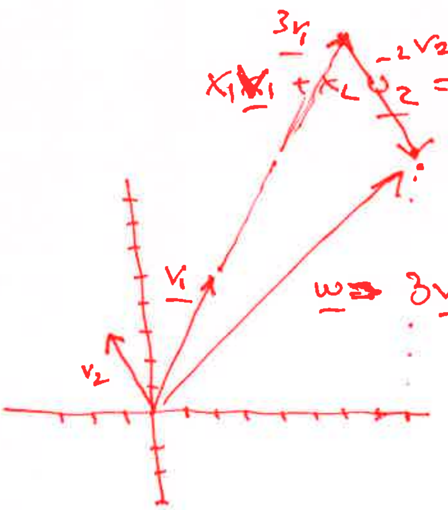
$$= \lambda \cdot A\underline{x}_1 + (1-\lambda) A\underline{x}_2$$

$$= \lambda \cdot \underline{b} + (1-\lambda)\underline{b}$$

$$= \lambda \underline{b} + \underline{b} - \lambda \underline{b} = \underline{b}$$

3.1. $\underline{w} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$ $\underline{v}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$x_1 \underline{v}_1 + x_2 \underline{v}_2 = \underline{w}$: $x_1 \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} + x_2 \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$



$\Downarrow$

$$\left(\begin{array}{cc|c} 2 & -1 & 8 \\ 5 & 3 & 9 \end{array}\right) \xrightarrow{-5/2} \left(\begin{array}{cc|c} 2 & -1 & 8 \\ 0 & 11/2 & -11 \end{array}\right) \cdot 2$$

$$\to \left(\begin{array}{cc|c} 2 & -1 & 8 \\ 0 & 11 & -22 \end{array}\right)$$

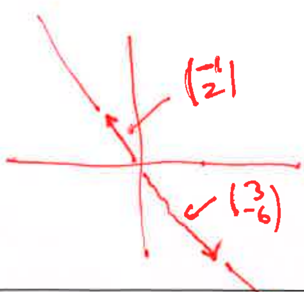
$2x_1 - x_2 = 8$ $\underline{x_1 = 3}$

$11x_2 = -22$ $\underline{x_2 = -2}$

$$\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 8 + 3 \neq 0$$

3.2

a)



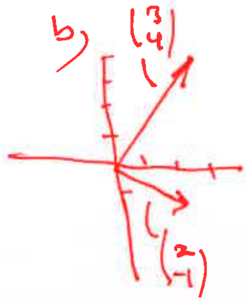
vectors along the same line

$\Downarrow$

linearly dependent

$$\begin{pmatrix} 3 \\ -6 \end{pmatrix} = -3 \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

b)



vectors not along same line

$\Downarrow$

vectors lin. independent

3.5 $\underline{v}_1, \ldots, \underline{v}_n$ m-vectors

$\left\{ \text{at least one vector is a linear comb. of the others} \right\} \Longleftrightarrow \left\{ x_1\underline{v}_1 + x_2\underline{v}_2 + \ldots + x_n \cdot \underline{v}_n = \underline{0} \text{ has non-trivial solution } (\underline{x} \neq \underline{0}) \right\}$

Assume: $\Longrightarrow$

$\underline{v}_1 = c_2 \cdot \underline{v}_2 + c_3 \cdot \underline{v}_3 + \ldots + c_n \cdot \underline{v}_n$

$\Downarrow$

$1 \cdot \underline{v}_1 - c_2 \cdot \underline{v}_2 - \ldots - c_n \cdot \underline{v}_n = \underline{0}$

$(1, -c_2, -c_3, \ldots, -c_n)$ is a solution of

there is a non-trivial sol.

Similar if one of the other vectors is a lin. comb.

$\Longleftarrow$

$x_1\underline{v}_1 + \ldots + x_n\underline{v}_n = \underline{0}$ has non-trivial sol. $(c_1, \ldots, c_n) \neq (0, 0, \ldots, 0)$

Assume: $c_1 \neq 0$

$c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n = \underline{0}$

$c_1\underline{v}_1 = -c_2\underline{v}_2 - c_3\underline{v}_3 - \ldots - c_n\underline{v}_n$

$\Downarrow$

$\underline{v}_1 = -\frac{c_2}{c_1} \cdot \underline{v}_2 - \frac{c_3}{c_1}\underline{v}_3 - \ldots - \frac{c_n}{c_1}\underline{v}_n$

3.12.

i) Null (A^TA) = Null (A):

Null (A^TA):

Sol. of $A^TA \cdot \underline{x} = \underline{0}$

Null (A):

Sol of $A\underline{x} = \underline{0}$

$\underline{x}$ sol. of $A\underline{x} = 0$

$\Downarrow$

$A^TA\underline{x} = A^T \cdot \underline{0} = \underline{0}$

$\Downarrow$

$\underline{x}$ sol. of $A^TA \cdot \underline{x} = \underline{0}$

$\Leftarrow$ ok.

$\underline{x}$ sol. of $A^TA \cdot \underline{x} = \underline{0}$

$\Downarrow$

$x^T \cdot A^TA\underline{x} = x^T \cdot \underline{0} = \underline{0}$

$(A\underline{x})^T \cdot (A\underline{x}) = \underline{0}$

Call $A\underline{x} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$:

$(h_1 \dots h_n) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = 0$

$h_1^2 + h_2^2 + \dots + h_n^2 = 0$

$\Downarrow$

$h_1 = h_2 = \dots = h_n = 0$

$\Downarrow$

$A\underline{x} = 0$.

$\Rightarrow$ ok.

3.14. $\underline{w} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} +1 \\ -1 \\ -2 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}$ $\quad \underline{v}_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$

$$x_1 \underline{v}_1 + x_2 \underline{v}_2 + x_3 \underline{v}_3 = \underline{w} \Rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right)$$

There are solutions

$\Updownarrow$

$h-5=0$

$\Updownarrow$

$\underline{h=5}$ (c) is correct.

4.3b

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\left((2-\lambda)(-2-\lambda)+2\right) - 1\cdot(-2) = 0$$

$$(1-\lambda)(\lambda^2-2)+2 = 0$$

$$-\lambda^3+\lambda^2+2\lambda = 0$$

$$-\lambda(\lambda^2-\lambda-2) = 0$$

$$\underline{\lambda=0} \text{ or } \lambda = \frac{1 \pm \sqrt{1+4\cdot 2}}{2} = \frac{1\pm 3}{2}$$

$$\underline{\lambda=2}, \ \underline{\lambda=-1}$$

Eigenvalues:

$\underline{\lambda = 0, 2, -1}$

E_0: $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ \cancel{0} & \cancel{-1} & \cancel{-1} \end{pmatrix}$

$2x+y-z=0$
$y+z=0$
z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -z \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

E_2: $\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 2 & 0 & -4 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & -4 \\ 0 & -1 & 1 \\ \cancel{0} & \cancel{1} & \cancel{-1} \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ z \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

E_{-1}: $\begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ \cancel{0} & \cancel{-2} & \cancel{-1} \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z/2 \\ -z/2 \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

4.4. A invertible, λ eigenvalue

$\Downarrow$

$A\underline{v} = \lambda \underline{v}$, A^{-1} exists $\Rightarrow$ $A^{-1} A \underline{v} = A^{-1} \lambda \underline{v}$

$\underline{v} = \lambda \cdot A^{-1} \underline{v}$

$\frac{1}{\lambda} \underline{v} = A^{-1} \underline{v}$

$\Downarrow$

$1/\lambda$ eigenvalue of A^{-1}

$\lambda \neq 0$ since $|A| =$ product of eigenvalues $\neq 0$

4.5

$$\begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -15 \\ -5 \\ 0 \end{pmatrix} = -5 \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{\lambda_1 = -5}$$

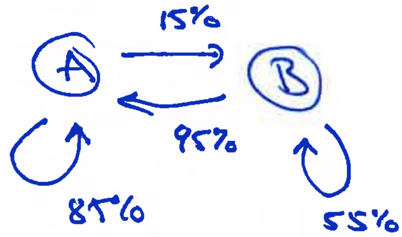
$$\text{"} \quad \cdot \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \\ -5 \end{pmatrix} = -5 \cdot \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \qquad \underline{\lambda_2 = -5}$$

$$\text{"} \quad \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ -5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \qquad \underline{\lambda_3 = 5}$$

$$D = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad P = \begin{pmatrix} -3 & -5 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

4.8

$$T = \begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix}$$



$$\underline{s} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

a) $$T \cdot \underline{s} = \begin{pmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{pmatrix} \cdot \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.53 \\ 0.47 \end{pmatrix}$$

↑ market shares after 1 year

b) $$\begin{vmatrix} 0.85-\lambda & 0.45 \\ 0.15 & 0.55-\lambda \end{vmatrix} = 0$$

$$(0.85-\lambda)(0.55-\lambda) - 0.45 \cdot 0.15 = 0$$

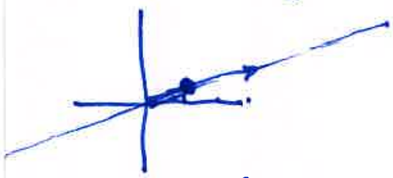
$$\lambda^2 - 1.4\lambda + 0.4 = 0$$

$$\underline{\lambda = 1}, \quad \lambda = \underline{0.40}$$

c) Find P s.t. $P^{-1}TP = D = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$

Eigenvectors:

$\lambda = 1$: $$\begin{pmatrix} -0.15 & 0.45 \\ \cancel{0.15} & \cancel{-0.45} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ $-0.15x + 0.45y = 0$, y free



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$$ $3y + y = 1$, $\underline{y = 1/4}$

$\lambda = 0.4$: $$\begin{pmatrix} 0.45 & 0.45 \\ \cancel{0.15} & \cancel{0.15} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ $0.45x + 0.45y = 0$, y free

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$ $$\underline{P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}}$$

We have: $P^{-1}TP = D$ $\quad P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$T^n = ?$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$

$P \cdot | \quad P^{-1}TP = D$

$TP = PD \quad | \cdot P^{-1}$

$T = \underline{PDP^{-1}}$

$\to T^n = (PDP^{-1})^n$

$= (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}}) \dots (\cancel{P}DP^{-1})$

$= \underline{\underline{PD^nP^{-1}}}$

d) $D^n = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix} \cdot \dots \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$

$= \begin{pmatrix} 1^n & 0 \\ 0 & 0.4^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.4^n \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ when $n \to \infty$.

$T^n = P \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.4^n \end{pmatrix} \cdot P^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.4^n \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$\to \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}$

$= \underline{\underline{\begin{pmatrix} 3/4 & 3/4 \\ 1/4 & 1/4 \end{pmatrix}}}$

Equilibrium: $\underbrace{T^n \cdot \underline{s}}_{\text{state after } n \text{ years}} \to \begin{pmatrix} 3/4 & 3/4 \\ 1/4 & 1/4 \end{pmatrix} \cdot \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}}}$

BREAK

4.9.
$$\begin{vmatrix} 4-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(\lambda^2-9\lambda+18)=0$$

$$\underline{\lambda=3},\ \underline{\lambda=3},\ \underline{\lambda=6}$$

E_3: $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ \cancel{1} & \cancel{1} & \cancel{2} \end{pmatrix}$

2 free vars

$\Downarrow$

$\dim E_3 = 2 =$ mult. of $\lambda=2$

A is diagonalizable

E_3: $x+y+2z=0$, y,z free $\Big\}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-2z \\ y \\ z \end{pmatrix} = y\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\cdot\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

E_6: $\begin{pmatrix} -2 & +1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad P^{-1} = \frac{1}{3}\begin{pmatrix} 0 & -1 & 1 \\ 3 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}^T = \frac{1}{3}\begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow A^{17} = (PDP^{-1})^{17}$$
$$= (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$$
$$= P\cdot D^{17}\cdot P^{-1}$$

$$A^{17} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 3^{17} & 0 & 0 \\ 0 & 3^{17} & 0 \\ 0 & 0 & 6^{17} \end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\cdot\begin{pmatrix} -3^{17} & -2\cdot 3^{17} & 6^{17} \\ 3^{17} & 0 & 0 \\ 0 & 3^{17} & 6^{17} \end{pmatrix}\cdot\begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2\cdot 3^{17}+6^{17} & -1\cdot 3^{17}+6^{17} & -2\cdot 3^{17}+2\cdot 6^{17} \\ 0 & 3\cdot 3^{17} & 0 \\ -3^{17}+6^{17} & -3^{17}+6^{17} & 3^{17}+2\cdot 6^{17} \end{pmatrix}$$

4.10. $A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

a) $|A| = s \cdot 6 = \underline{\underline{6s}}$

$$\operatorname{rk} A = \begin{cases} 3, & s \neq 0 \leftarrow |A| = 6s \neq 0 \\ \underline{2}, & s = 0 \leftarrow M_{13,13} = \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 6 \neq 0 \end{cases}$$

b) $$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 0 & s-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$(s-\lambda)(\lambda^2 - 5\lambda + 6) = 0 \quad \underline{\lambda = s}$ or $\underline{\lambda = 2}, \underline{\lambda = 3}$

$A\underline{v} = \begin{pmatrix} 6 \\ s \\ 6 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \lambda = 6$ if $s = 6$

$\underline{v}$ is an eigenvector if $\underline{\underline{s=6}}$, but not otherwise

c) $\underline{\lambda = s, 2, 3:}$

$s \neq 2,3$: three eigenvalues of $m=1$ $\Rightarrow$ A $\underline{\text{diagonalizable}}$

$\underline{s=2}$: $\lambda = 2,2,3$

$\underline{\lambda = 2 \text{ (mult}=2):}$ $\begin{pmatrix} -1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 7 & -2 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

one free var

dim $E_2 \neq m \Rightarrow$ A $\underline{\underline{\text{not diag.}}}$

$\underline{s=3}$: $\lambda = 2,3,3$

$\underline{\lambda = 3 \text{ (mult } 2):}$ $\begin{pmatrix} -2 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

one free var.

dim $E_3 \neq m \Rightarrow$ A $\underline{\underline{\text{not diag}}}$

$\underline{\text{Concl:}}$ A diagonalizable for $s \neq 2,3$

4.11. $A = \begin{pmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{pmatrix}$

a) $|A| = (t+2)\cdot 5 - (t-8)\cdot(-5)$
$= 5(t+2) + 5(t-8) = \underline{10t-30}$

$|A| \neq 0$

$\operatorname{rk} A = \begin{cases} 3, & t \neq 3 \\ 2, & t = 3 \end{cases}$

$M_{12,12} = \begin{vmatrix} 1 & 1 \\ 0 & 5 \end{vmatrix} = 5 \neq 0$

b) $\begin{vmatrix} 1-\lambda & 1 & -4 \\ 0 & t+2-\lambda & t-8 \\ 0 & -5 & 5-\lambda \end{vmatrix} = 0$

$(1-\lambda)\cdot\left((t+2-\lambda)(5-\lambda) - (-5)(t-8)\right) = 0$

$(1-\lambda)\cdot\left(\lambda^2 - (t+7)\lambda + (10t-30)\right) = 0$

$\operatorname{tr}\begin{pmatrix} t+2 & t-8 \\ -5 & 5 \end{pmatrix}$ $\quad \begin{vmatrix} t+2 & t-8 \\ -5 & 5 \end{vmatrix} = |A|$

$\underline{\lambda = 1}$ or $\lambda^2 - (t+7)\lambda + 10(t-3) = 0$

$$\lambda = \frac{(t+7) \pm \sqrt{(t+7)^2 - 4\cdot 1\cdot 10(t-3)}}{2\cdot 1}$$

$$= \frac{(t+7) \pm \sqrt{t^2+14t+49-40t+120}}{2}$$

$(t-13)^2$

$$= \frac{(t+7) \pm \sqrt{t^2-26t+169}}{2} = \frac{(t+7) \pm (t-13)}{2}$$

$\lambda = \frac{2t-6}{2} = \underline{t-3}$ or $\lambda = \frac{20}{2} = \underline{10}$

Eigenvalues:
$\lambda = 1,\ t-3,\ 10$

c) Eigenvalues with multiplicity: $t-3=1$ or $t-3=10$

$\underline{t=4}$ or $\underline{t=13}$

$\underline{t \neq 4, 13}$: three distinct eigenvalues $\Rightarrow$ A diag.

$\underline{t=4}$: $\lambda = 1, 1, 10$

$\underline{\lambda = 1 \ (m=2)}$:
$$\begin{pmatrix} 0 & 1 & -4 \\ 0 & 5 & -4 \\ 0 & -5 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -4 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{pmatrix}$$

one free var

$\dim E_1 = 1 \neq 2 \Rightarrow$ A $\underline{\underline{\text{not diag.}}}$

$\underline{t=13}$: $\lambda = 1, 10, 10$

$\underline{\lambda = 10}$ $(m=2)$:
$$\begin{pmatrix} -9 & 1 & -4 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{pmatrix}$$

one free var

$\dim E_{10} = 1 \neq m \Rightarrow$ A $\underline{\underline{\text{not diag.}}}$

A diag. for $\underline{\underline{t \neq 4, 13}}$

C Midterm 10/2017:

1 4×6 linear system : $A \cdot \underline{x} = \underline{b}$

pivot position in every row of A

$$\begin{array}{ccccccc} & & \downarrow & & & \downarrow & \\ \end{array}$$
$$\left(\begin{array}{cccccc|c} \circ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \circ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \circ & \cdot & \cdot \end{array}\right)$$

two free var.

D

2 $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & t \end{pmatrix} = \left(\underline{v_1} \,|\, \underline{v_2} \,|\, \underline{v_3} \right)$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ lin. indep.
$\Updownarrow$
$|A| \neq 0$

$|A| = 1 \cdot (2t-12) - 1\,(t-3) + 1 \cdot 2$
$= t - 7$

lin indep. for $t \neq 7$ $\Leftrightarrow$ lin depend. for $t = 7$

B

all 2-minors are zero

$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

3. $A = \begin{pmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{pmatrix}$

$\text{rk}\, A = 3 \Leftrightarrow |A| \neq 0$
$\text{rk}\, A < 3 \Leftrightarrow |A| = 0$

$t=1$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\text{rk} = 1$

$t=-2$: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ $\text{rk} = 2$

$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \neq 0$
$\text{rk}\, A = 2$ for $t = -2$

$\begin{vmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = \underline{t(t^2-1)} - 1 \cdot (t-1) + 1 \cdot (1-t) = 0$

$t(t-1)(t+1) - (t-1) - (t-1) = 0$

$(t-1)(t(t+1) - 2) = 0$

$\underline{t = 1}$ or $t^2 + t - 2 = 0$

$\underline{t = -2}, \ \underline{t = 1}$

$$\text{rk}\, A = \begin{cases} 3, & t \neq 1, -2 \\ 2, & t = -2 \\ 1, & t = 1 \end{cases}$$

A

4.

$$A = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 2 & 0 \\ 1 & 3 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ 0 & 2-\lambda & 0 \\ 1 & 3 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\cdot \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)\cdot(\lambda^2-7\lambda+10) = 0$$

$\underline{\lambda = 2}$ $\qquad \lambda = \dfrac{7 \pm \sqrt{49-40}}{2} = \dfrac{7 \pm 3}{2}$

$\underline{\lambda = 5}$ $\quad \underline{\lambda = 2}$

$\lambda = 2$ mult 2

$\lambda = 5$ mult 1 $\qquad$ (B)

5.

$$A = \begin{pmatrix} 1 & s & 1 \\ 0 & 1 & s \\ 0 & 0 & -1 \end{pmatrix} \qquad \underbrace{\underline{\lambda_1 = 1} \quad \underline{\lambda_2 = 1}}_{m=2} \qquad \underbrace{\lambda_3 = -1}_{m=1}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

<u>Eigenvectors:</u>

$\underline{\lambda = 1}$:

(E_1)

$$\begin{pmatrix} 0 & s & 1 \\ 0 & 0 & s \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & s & 1 \\ 0 & 0 & -2 \\ 0 & 0 & s \end{pmatrix}$$

$\underline{s = 0}$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $\cdot 2$

2 free var. (x, y)

$\underline{s \neq 0}$: $\begin{pmatrix} 0 & s & 1 \\ 0 & 0 & -2 \\ 0 & 0 & s \end{pmatrix}$ $\cdot s/2$

1 free var. (x)

A diag $\Leftrightarrow$ $\underline{\underline{s = 0}}$ $\qquad$ (C)

$\lambda = -1$: Since mult. $m = 1$, there are enough eigenvectors $\Big\}$ $\dim E_{-1} = 1$, $m = 1$