Selected Problems:

(A) Key problems: 5.1c, 5.3d, 5.2e, 6.1d, 6.3bd, 6.4d, 7.2c

(B) Workbook problems: 5.14, 5.15, 6.1c, 6.2a

(C) Midterm problems: 10/2018 Problem 1–8, 10/2016 Pb. 3,5
05/2017 Pb. 8, 10/2014 Pb. 5

Extra Problem: Key problem 3.3

(A) Key problems

5.1 c)

$$A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$$

regular

$\lambda = 1$:

$$\begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix} \begin{matrix} \cdot 100 \\ \cdot 100 \\ \cdot 100 \end{matrix} \to \begin{pmatrix} 5 & 8 & -30 \\ 20 & -10 & 20 \\ -25 & 2 & 10 \end{pmatrix} \begin{matrix} -4 \\ \\ 5 \end{matrix}$$

$$\to \begin{pmatrix} 5 & 8 & -30 \\ 0 & -42 & 140 \\ \cancel{0} & \cancel{42} & \cancel{-140} \end{pmatrix}$$

$5x + 8y - 30z = 0$
$-42y + 140z = 0$
z free

$y = \frac{140z}{42} = \frac{20z}{6} = \frac{10}{3}z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3}z \\ 10/3 \cdot z \\ z \end{pmatrix} = z \begin{pmatrix} 2/3 \\ 10/3 \\ 1 \end{pmatrix}$$

$5x = -8y + 30z$
$= -8 \cdot \frac{10}{3}z + \frac{90}{3}z$
$= \frac{10z}{3}$

$x = \frac{1}{5} \cdot \frac{10}{3}z = \frac{2}{3}z$

$\frac{2}{3}z + \frac{10}{3}z + \frac{3}{3}z = 1$

$\frac{15z}{3} = 1 \quad z = \frac{3}{15}$

$x = \frac{2}{3} \cdot \frac{3}{15} = \frac{2}{15} \quad y = \frac{10}{3} \cdot \frac{3}{15} = \frac{10}{15} \quad z = \frac{3}{15}$

$$\underline{v} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}$$

5.3d $f(x,y,z,w) = xw - yz$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$

$D_1 = 0$ $\Delta_1 = 0, 0, 0, 0$

$D_2 = 0$ $\Delta_2 = 0, 0, -1/4, -1/4, 0, 0$

$D_3 = 0$

$D_4 = -\frac{1}{2} \cdot (-1/2)(0 + 1/4)$

$= 1/4 \cdot (+1/4) = +1/16$

$\Delta_2 = -1/4 < 0$ $\rightarrow$ indefinit

Note: A positive definite $\Leftrightarrow D_1, D_2, D_3, D_4 > 0 \Leftrightarrow \Delta_1, \Delta_2, \Delta_3, \Delta_4 > 0$

A positive semi defn. $\Leftrightarrow \Delta_1, \Delta_2, \Delta_3, \Delta_4 \geq 0$

Alt: $$A = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$

$\Delta_2^{23,23} = 0 - 1/4 = -1/4 < 0$

A indefinite

5.2e $$A = \begin{pmatrix} -1 & -2 & -2 \\ -2 & -4 & -4 \\ -2 & -4 & -2 \end{pmatrix}$$

$D_1 = -1$

$D_2 = 0$

$D_3 = 0$

$$\begin{pmatrix} -1 & -2 & -2 \\ 0 & 0 & 0 \\ -2 & -4 & -2 \end{pmatrix}$$

$\Delta_2^{23,23} = 8 - 16 = -8 < 0$

A indefinite

6.1(d) $f = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$D_1 = 1 > 0$

$D_2 = 1 - 1/4 = 3/4 > 0$

$D_3 = 1 \cdot 3/4 - 1/2(1/2 - 0) = 1/2 > 0$

$D_4 = -1/2 \cdot \begin{vmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{vmatrix} + 1 \cdot 1/2$

$= -\frac{1}{2}\left(\frac{1}{2} \cdot \frac{3}{4}\right) + \frac{1}{2}$

$= -\frac{3}{16} + \frac{8}{16} = \frac{5}{16} > 0$

positive defn.

6.3 b same function as in 6.1d.

$f'_x = 2x + y = 0$

$f'_y = 2y + x + z = 0$

$f'_z = 2z + y + w = 0$

$f'_w = 2w + z = 0$

$$H(f) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} = 2A$$

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$\underbrace{}_{2A}$

$(x, y, z, w) = (0,0,0,0)$ stationary pt.

since $|2A| = 2^4 \cdot |A| = 5 \neq 0$

$H(f)(0,0,0,0) = 2A$ pos. defn.

since A is positive defn.

$\Downarrow$

$(0,0,0,0)$ local min

Global:

f is convex since $H(f)(\underline{x})$ is pos. defn. at all pts $\underline{x}$.

$(0,0,0,0)$ global min

d) $f(x,y,z,w) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

$f'_x = -4x^3 - 4x + 6z = 0$
$f'_y = -6y = 0$
$f'_z = 6x - 12z = 0$

$$H(f) = \begin{pmatrix} -12x^2-4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

Stationary pts:

$-6y = 0 \Rightarrow y = 0$

$6x - 12z = 0 \Rightarrow x = 2z$

$-4x^3 - 4x + 6z = 0$

$-4(2z)^3 - 4 \cdot 2z + 6z = 0$

$-32z^3 - 8z + 6z = 0$

$2z(-16z^2 - 1) = 0$

$z = 0$ or $-16z^2 - 1 = 0$ impossible

$\Downarrow$

$(x,y,z) = (0,0,0)$

$$H(f)(0,0,0) = \begin{pmatrix} -4 & 0 & +6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

$D_1 = -4 < 0$
$D_2 = 24 > 0$
$D_3 = -6 \cdot (48 - 36)$
$= -6 \cdot 12 = -72 < 0$

neg. defn.

$\Downarrow$

$(0,0,0)$ local max

$$H(f) = \begin{pmatrix} -12x^2-4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

Is f concave?

$D_1 = -12x^2 - 4 < 0$ for all x,y,z
$D_2 = 72x^2 + 24 > 0$ — " —
$D_3 = -6((-12x^2-4)(-12) - 36)$
$= -6(144x^2 + 48 - 36)$
$= -6(144x^2 + 12) < 0$ — " —

f concave since H(f) is negative defn. for all $\underline{x}$

$(0,0,0)$ global max

64 d) same fn. as in the previous problem

f is concave

7.2 c) $f(x,y,z,w) = (x+y+z+w)^4 = u^4$, $u = x+y+z+w$

$f'_x = 4u^3 \cdot u'_x = 4u^3 \cdot 1$

$f'_y = 4u^3 \cdot u'_y = 4u^3$

$f'_z = 4u^3 \cdot u'_z = 4u^3$

$f'_w = 4u^3 \cdot u'_w = 4u^3$

$f''_{xx} = 12u^2 \cdot 1 = 12u^2$ $\quad f''_{xy} = 12u^2 \cdot 1 = 12u^2$

$$H(f) = \begin{pmatrix} 12u^2 & 12u^2 & 12u^2 & 12u^2 \\ \vdots & & & \end{pmatrix}$$

$$= 12u^2 \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{rk } H(f) = \begin{cases} 1, & u \neq 0 \\ 0, & u = 0 \end{cases}$$

$\leftarrow$ pos. and neg. semidef.

$(x+y+z+w=0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$D_1 = 12u^2 > 0$ when $u \neq 0$

rk = 1

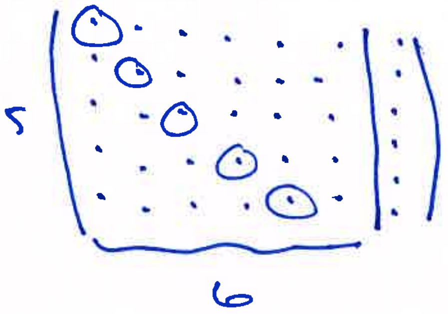
$\Downarrow$ RRC

pos. semidef.

f convex but not concave

Ex 10/2018.

1.



one free variable

(C)

2.

$$\underline{v}_1 = \begin{pmatrix} t \\ 2 \\ 3 \\ 5 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 3 \\ 6 \\ t \\ 9+t \end{pmatrix} \rightsquigarrow A = \begin{pmatrix} \boxed{\begin{matrix} t & 3 \\ 2 & 6 \end{matrix}} \\ \begin{matrix} 3 & t \\ 5 & 9+t \end{matrix} \end{pmatrix}$$

$\{\underline{v}_1, \underline{v}_2\}$ lin. independent

$\Updownarrow$

$\text{rk } A = 2$

$$\begin{vmatrix} t & 3 \\ 2 & 6 \end{vmatrix} = 6t - 6$$

$t = 1$: $M = 0$

$t \neq 1$: $M \neq 0$, $\text{rk} = 2$ lin indep.

$\underline{t=1}$:
$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \\ \boxed{\begin{matrix} 3 & 1 \\ 5 & 10 \end{matrix}} \end{pmatrix}$$

$$M_{34,12} = 30 - 5 = 25 \neq 0$$

lin indep.

(A)

3.

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 4 & 0 & 6 \\ t & -1 & 5 & 3 \end{pmatrix}$$

$$M = -2 \cdot (14) + 4 \cdot (5+t)$$
$$= -8 + 4t$$

$t \neq 2$: $M \neq 0$, $\text{rk} = 3$

$t = 2$: $M = 0$

(A)

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \\ 2 & -1 & 3 \end{vmatrix} = 1 \cdot (18) - 3 \cdot (-6) + 4 \cdot (-10)$$
$$= 36 - 40 = -4 \neq 0$$

$\text{rk} = 3$

4. $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ $\qquad \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$

$$(2-\lambda) \cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \cdot (\lambda^2 - 6\lambda + 8) = 0$$

$$(2-\lambda)\ (\lambda-2)(\lambda-4) = 0$$

$$\lambda_1 = \lambda_2 = 2 \qquad \lambda_3 = 4$$

(B)

5. A symmetric $\Rightarrow$ A diagonalizable (A)

6. $A = \begin{pmatrix} 0.4 & 0.2 & 0.10 \\ 0.4 & 0.6 & 0.10 \\ 0.2 & 0.2 & 0.80 \end{pmatrix}$ $\qquad \begin{pmatrix} -0.6 & 0.2 & 0.1 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.2 \end{pmatrix} \cdot 10$

regular

$$\to \begin{pmatrix} 2 & 2 & -2 \\ 4 & -4 & 1 \\ -6 & 2 & 1 \end{pmatrix} \begin{matrix} \\ \cdot(-2) \\ 3 \end{matrix} \to \begin{pmatrix} 2 & 2 & -2 \\ 0 & -8 & 5 \\ \cancel{0} & \cancel{8} & \cancel{-5} \end{pmatrix}$$

$2x + 2y - 2z = 0$ $\qquad x = -y + z = -\frac{5}{8}z + \frac{8}{8}z = \frac{3}{8}z$

$-8y + 5z = 0$ $\qquad y = \frac{5}{8}z$

z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/8 \cdot z \\ 5/8 \cdot z \\ z \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} 3/16 \\ 5/16 \\ 8/16 \end{pmatrix}$$

$v_2 = 5/16 = 0.3125$ (D)

$$3/8\, z + 5/8\, z + z = 1$$

$$2z = 1$$

$$z = 1/2$$

7.

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & 2 \\ -2 & 2 & 8 \end{pmatrix}$$

$D_1 = 3 > 0$

$D_2 = 9-4 = 5 > 0$

$D_3 = 3\cdot 20 - 2\cdot 20 - 2\cdot 10$

$= 60 - 40 - 20 = \underline{0}$

$\underline{\text{rk}\, A = 2}$ $\Downarrow$ RRC

positive semidefn.

$\underline{\underline{\text{not}}}$ pos. defn.

A

8.

$f(x,y,z) = 1 - (x-y+z)^4 = 1 - u^4$, $u = x-y+z$

$f'_x = -4u^3 \cdot 1 = -4u^3$

$f'_y = -4u^3 \cdot (-1) = 4u^3$

$f'_z = -4u^3 \cdot 1 = -4u^3$

$$H(f) = \begin{pmatrix} -12u^2 & 12u^2 & -12u^2 \\ 12u^2 & -12u^2 & 12u^2 \\ -12u^2 & 12u^2 & -12u^2 \end{pmatrix}$$

$(x,y,z) = (1,1,0)$: $\underline{u(1,1,0) = 0}$

$$H(f)(1,1,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

no conclusion from second derivative test.

a) $\underline{\text{From Defn.:}}$

$f(1,1,0) = 1$

$f(x,y,z) \leq 1$ forall (x,y,z)

$\Downarrow$

$(1,1,0)$ local and global $\underline{\underline{\text{max.}}}$

b) $\underline{\text{Is } f \text{ convex / concave?}}$

$$H(f) = \begin{pmatrix} -12u^2 & 12u^2 & -12u^2 \\ 12u^2 & -12u^2 & 12u^2 \\ -12u^2 & 12u^2 & -12u^2 \end{pmatrix}$$

$D_1 = -12u^2 \leq 0$

$D_2 = 0$

$D_3 = 0$

$\text{rk}\, H(f) \leq 1$

$\text{rk}\, H(f) = \begin{cases} 1, & u \neq 0 \\ 0, & u = 0 \end{cases}$

$\text{rk} = 1$, $D_1 < 0$, RRC: $\underline{\text{neg. semidefn.}}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Downarrow$

f $\underline{\text{concave}}$

$(1,1,0)$ is global max.

D

Midterm 10/2016

5) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

not symmetric
upper triangular $\Rightarrow \lambda_1 = \lambda_2 = 1, \ \lambda_3 = 5$

From Theory:

n eigenvalues of $m=1$ $\Big\} \Rightarrow$ A diagonalizable

if there are eigenvalue λ with $m>1$ $\Big\}$ A diagonalizable $\Updownarrow$ $\dim E_\lambda = m$ (# free var's $= m$)

a) $s=1$: $m=3$ at $\lambda=1$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow s=1$ subtract $\lambda=1$

$\dim E_1 = 2 < m = 3$

b) $s \neq 1$: $m=2$ at $\lambda=1$, $m=1$ at $\lambda=s$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s-1 \end{pmatrix}$ $\dim E_1 = 1 < m = 2$

a) $s=1$: $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $P = (v_1 | v_2 | ?)$ not diag.

b) $s \neq 1$: $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix}$ $P = (v_1 | ? | v_3)$ not diag.

Midterm 05/2017

(8) $f(x,y) = \sqrt{x^2+y^2+3} = \sqrt{u}$, $u = x^2+y^2+3$

$= u^{1/2}$

$$f'_x = \boxed{\tfrac{1}{2} u^{-1/2}} \cdot u'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{\cancel{2}x}{\cancel{2}\sqrt{u}} = \frac{x}{\sqrt{u}}$$

$$= x \cdot u^{-1/2}$$

$$f''_{xx} = (x \cdot u^{-1/2})'_x = 1 \cdot u^{-1/2} + x \cdot (-\tfrac{1}{\cancel{2}}) u^{-3/2} \cdot \cancel{2}x$$

$$= \frac{u \cdot 1}{u \cdot \sqrt{u}} - \frac{x^2}{u\sqrt{u}} = \frac{u - x^2}{u\sqrt{u}} = \frac{y^2+3}{u\sqrt{u}}$$

$$f''_{xy} = (x \cdot u^{-1/2})'_y$$

$$= x \cdot (-\tfrac{1}{\cancel{2}}) u^{-3/2} \cdot \cancel{2}y = -\frac{xy}{u\sqrt{u}}$$

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix}$$

$D_1 = f''_{xx} = \frac{y^2+3}{u\sqrt{u}} > 0$

for all x,y

$$= \begin{pmatrix} \frac{y^2+3}{u\sqrt{u}} & \frac{-xy}{u\sqrt{u}} \\ \frac{-xy}{u\sqrt{u}} & \frac{x^2+3}{u\sqrt{u}} \end{pmatrix}$$

$$D_2 = \frac{(y^2+3)(x^2+3)}{u^2 \cdot u} - \frac{x^2y^2}{u^2 \cdot u}$$

$$= \frac{\cancel{x^2y^2} + 3x^2 + 3y^2 + 9 \cancel{- x^2y^2}}{u^3} > 0$$

for all x,y

(B) f convex

$$f''_{yy} = \left(\frac{y}{\sqrt{u}}\right)'_y = (y \cdot u^{-1/2})'_y = 1 \cdot u^{-1/2} + y \cdot (-\tfrac{1}{\cancel{2}}) u^{-3/2} \cdot \cancel{2}y$$

$$= \frac{u \cdot 1}{u \cdot \sqrt{u}} - \frac{y^2}{u\sqrt{u}} = \frac{u - y^2}{u\sqrt{u}} = \frac{x^2+3}{\sqrt{u} \cdot u}$$