Plan

1 Workbook Problems: 7.1f, 7.4, 7.9, 7.11, 8.7, 8.12, 8.13
2 Key Problems: 7.1d, 7.4b, 8.2d, 8.3b, 9.1, 9.3
3 Exam Problems: Final exam 01/2017 3-4, 11/2017 3-4

Final exam 01/2017, Pb. 4

$$\max f = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2 \quad \text{when} \quad x+y-z \geq 2$$

$\boxed{\text{not bounded}}$

a) Kuhn-Tucker conditions:

Same as before

Std. form: $\max f(x,y,z) =$ when $-x-y+z \leq -2$

$$L = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2 - \lambda(-x-y+z)$$

FOC:
$$\begin{cases} L'_x = -4x + 2y - 2z + \lambda = 0 \\ L'_y = 2x - 4y + 4z + \lambda = 0 \\ L'_z = -2x + 4y - 4z - \lambda = 0 \end{cases}$$

C: $x+y-z \geq 2$

CSC: $\lambda \geq 0$ and $\lambda(x+y-z-2) = 0$

b) Solve the KT problem:

FOC:
$$\left(\begin{array}{cccc|c} -4 & 2 & -2 & 1 & 0 \\ 2 & -4 & 4 & 1 & 0 \\ -2 & 4 & -4 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 2 & -4 & 4 & 1 & 0 \\ -4 & 2 & -2 & 1 & 0 \\ -2 & 4 & -4 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 2 & -4 & 4 & 1 & 0 \\ 0 & -6 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$2x - 4y + 4z + \lambda = 0$$
$$-6y + 6z + 3\lambda = 0$$

i) $x+y-z=2$

$\lambda \geq 0$

$$\left(\begin{array}{cccc|c} 2 & -4 & 4 & 1 & 0 \\ 0 & -6 & 6 & 3 & 0 \\ 1 & 1 & -1 & 0 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & -6 & 6 & 3 & 0 \\ 2 & -4 & 4 & 1 & 0 \end{array}\right) \cdot (-2)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & -6 & 6 & 3 & 0 \\ 0 & -6 & 6 & 1 & -4 \end{array}\right) \cdot (-1)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & -6 & 6 & 3 & 0 \\ 0 & 0 & 0 & -2 & -4 \end{array}\right)$$

$-2\lambda = -4$ $\quad \underline{\underline{\lambda = 2}}$ $\quad$ ($\lambda \geq 0$ ok)

$-6y + 6z + 3\cdot 2 = 0$

$-6y = -6z - 6$

$y = \underline{\underline{z+1}}$

$x + y - z = 2$

$x = 2 - (z+1) + z = \underline{\underline{1}}$

$(x,y,z;\lambda) = \underline{(1, z+1, z; 2)}$

cand. pts.

ii) $x+y-z > 2$

$\lambda = 0$

SOC: $\lambda = 2$

$h(x,y,z) = \mathcal{L}(x,y,z;2)$

$= \cancel{-3} - 2x^2 + 2xy - 2xz - 2y^2$
$+ 4yz - 2z^2 \cancel{- 2(-x-y+z)}$

$$H(h) = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}$$

$D_1 = -4 < 0$

$D_2 = 12 > 0$

$D_3 = \cancel{\ldots - 4\cdot 12}$

$\cancel{\ldots - 48 = -32 < 0}$

$\cancel{H(h)} = -2\cdot 0 - 4(-12)$
$- 4\cdot 12 = 0$

$\text{rk } H(h) = 2 \Rightarrow$ RCC $\quad H(h)$ neg. semidef.

h concave $\Leftarrow$

$\Downarrow$ SOC

$\underline{\underline{(1, z+1, z) \text{ max}}}$ $\quad$ (inf. many max. pts)

Max value:

$f(1,1,0) = \underline{\underline{-5}}$

$-x-y+z \leq -2$

c) max $f(x,y,z) = \dots$ when $ax - y + z \leq -2$
(same as before)

$$\boxed{\underline{a=-1}: \quad f^*(-1) = -5, \quad \underline{x}^*(-1) = (1, z+1, z), \quad \lambda^*(-1) = 2}$$

$\mathcal{L} = f(x,y,z) - \lambda(ax - y + z)$
$= \dots - \lambda a x$

Envelope Thm.:

$$\boxed{\frac{df^*(a)}{da} = \mathcal{L}'_a(\underline{x}^*(a); \lambda^*(a)) = -\lambda^*(a)x^*(a) = -\lambda^*(a) \cdot x^*(a)}$$

Estimate: $f^*(-1.12) \cong f^*(-1) + \Delta a \cdot \dfrac{df^*(a)}{da}$

$= -5 + (-0.12) \cdot (-2 \cdot 1)$

$= -5 + 0.24 = \underline{\underline{-4.76}}$

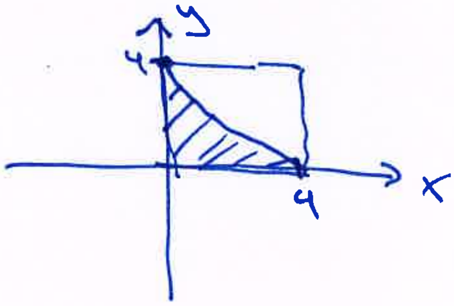
Workbook

7.1 d) $\{(x,y) : \sqrt{x}+\sqrt{y} \le 2\}$

Closed: Yes ($\le$)

Bounded: Yes

$$\boxed{\begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le 4 \end{array}}$$



Convex:

Is $D = \{(x,y) : \sqrt{x}+\sqrt{y} \le 2\}$ a convex ?

Defn: D is convex

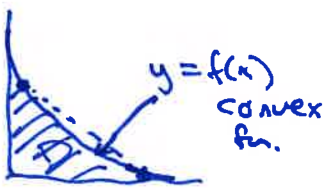
$\Updownarrow$

P, Q in $D \Rightarrow [P,Q]$ is contained in D

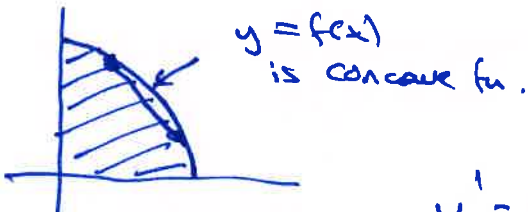
$$\sqrt{x}+\sqrt{y} = 2$$
$$\sqrt{y} = 2-\sqrt{x}$$
$$y = (2-\sqrt{x})^2$$
$$= 4 - 4\sqrt{x} + x$$

See that:

$x \ge 0 \,,\, y \ge 0$



D is not convex



D is convex

$$y' = -4 \cdot \tfrac{1}{2} x^{-1/2} + 1$$
$$= -2 \cdot x^{-1/2} + 1$$
$$y'' = x^{-3/2} = \frac{1}{x^{3/2}}$$
$$y'' > 0$$

y convex

Conclusion:

D is not convex

7.4. max $f(x,y,z) = xyz$ when $\begin{cases} x^2+y^2=1 \\ x+z=1 \end{cases}$

$$\mathcal{L} = xyz - \lambda_1(x^2+y^2) - \lambda_2\cdot(x+z)$$

$$L'_x = yz - \lambda_1\cdot 2x - \lambda_2 = 0$$

$$L'_y = xz - \lambda_1\cdot 2y = 0 \qquad \lambda_1 = \frac{xz}{2y}$$

$$L'_z = xy - \lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = xy$$

$\boxed{\text{check } y=0}$

$$x^2+y^2=1 \Rightarrow y^2 = 1-x^2$$

$$x+z=1 \Rightarrow z = 1-x$$

$$yz - \cancel{2}\cdot\frac{xz}{\cancel{2}y}\cdot x - xy = 0 \quad |\cdot y$$

$$y^2z - x^2z - xy^2 = 0$$

$$(1-x^2)\cdot\underline{(1-x)} - x^2\,\underline{(1-x)} - x\cdot\underbrace{\underline{(1-x^2)}}_{(1-x)(1+x)} = 0$$

$$(1-x)\cdot\left[1-x^2-x^2-x(1+x)\right] = 0$$

$x=1$ or $-3x^2-x+1=0$

$y=0$

$z=0$

$\lambda_1=0$

$\lambda_2=0$

$-2\lambda_1-\lambda_2=0$

$\lambda_2=0$

$(1,0,0;0,0)$

$$x = \frac{1\pm\sqrt{1+12}}{-6} = \frac{1\pm\sqrt{13}}{-6}$$

$x = \frac{1+\sqrt{13}}{-6}$ or $x = \frac{1-\sqrt{13}}{-6}$

two points

two points

8.7 max xy when $x^2+y^2 \le 1$ std. form.

$$L = xy - \lambda(x^2+y^2)$$

FOC: $L'_x = y - \lambda \cdot 2x = 0$
$L'_y = x - \lambda \cdot 2y = 0$

C: $x^2+y^2 \le 1$ bounded

CSC: $\lambda \ge 0$ and
$\lambda \cdot (x^2+y^2-1) = 0$

$\Downarrow$

$y = 2\lambda x$
$x - 2\lambda \cdot (2\lambda x) = 0$
$x - 4\lambda^2 \cdot x = 0$
$x(1-4\lambda^2) = 0$
$x = 0$ or $\lambda^2 = 1/4$
$\lambda = 1/2$ or $\lambda = -1/2$

$x=0$:
$y = 0$
$\lambda = 0$
$(0,0;0)$
$f = 0$

$\lambda = 1/2$:
$y = x$
$x^2+y^2 = 1$
$2x^2 = 1$
$x^2 = 1/2$
$x = \pm\sqrt{1/2}$
$y = x$
$(\sqrt{1/2}, \sqrt{1/2}; 1/2)$
$(-\sqrt{1/2}, -\sqrt{1/2}; 1/2)$ $f = 1/2$

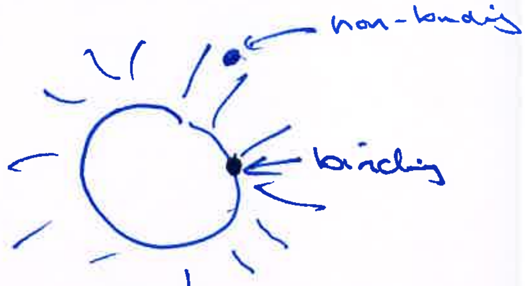
$\lambda = -1/2$
not possible $\lambda \ge 0$

Conclusion:
bounded $\underset{EVT}{\Rightarrow}$ there is a max

NDCQ fails: no points
i) $x^2+y^2=1$:
$J = (2x \;\; 2y)$
$rk\, J = 1$
$\Downarrow$
fails if $x=y=0$
not adm. $x^2+y^2 = 0 \ne 1$
ii) $x^2+y^2<1$:
no condition

Conclusion: $f_{max} = 1/2$
at $(\sqrt{1/2}, \sqrt{1/2})$, $(-\sqrt{1/2}, -\sqrt{1/2})$

Key Problem 8.2



b) $3x^2+3y^2+8z^2 \geq 1$

Admissible pts such that NDCQ fails: **None**

Binding:

$3x^2+3y^2+8z^2=1$: $\text{rk } J = \begin{pmatrix} 6x & 6y & 16z \end{pmatrix} < 1$

$6x = 6y = 16z = 0$

$\cancel{(0,0,0)}$ not adm and binding

Non-binding:

$3x^2+3y^2+8z^2 > 1$

$\cancel{\text{rk } (J = \begin{pmatrix} 6x & 6y & 16z \end{pmatrix}) \leq 1}$

no condition

d) $xy - zw = 1$
$x+y+z+w = 4$

$$J = \begin{pmatrix} y & x & -w & -z \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

NDCQ: $\text{rk } J = 2$

NDCQ fails: $\text{rk } J < 2$

$\Updownarrow$

all 2-minors $= 0$

$y-x=0 \quad y=x$
$x+w=0 \quad w=-x$
$-w+z=0 \quad z=w=-x$

$(x,y,z,w) = (x,x,-x,-x)$

$y+w=0$ ok
$x+z=0$ ok.

NDCQ fails: $(x,x,-x,-x)$

$$\begin{pmatrix} x & x & x & x \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$x^2 - x^2 = 1$
$0 = 1$
not possible

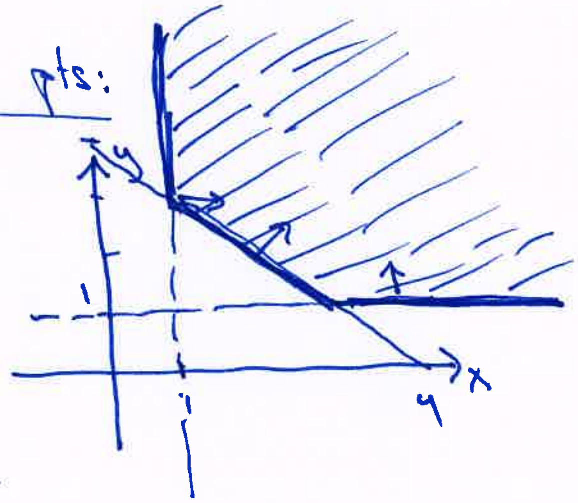
Conclusion:
no adm. pts where NDCQ fails.

Workbook, 8.13

$$\max \; f(x,y) = \ln(x^2 y) - x - y \quad \text{when} \quad \begin{cases} x+y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$$

a) Sketch the set of adm. pts:

$$D: \begin{cases} x+y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$$



b) Solve KT problem:

Std. form: $\max \; f(x,y) \quad \text{when} \quad \begin{cases} -x-y \leq -4 \\ -x \leq -1 \\ -y \leq -1 \end{cases}$

$x \geq 1$: boundary $x = 1$

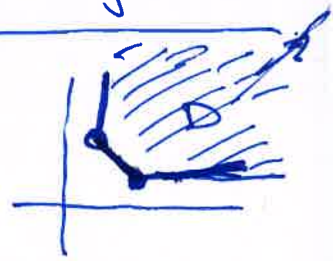
$y \geq 1$: " $y = 1$

$x+y \geq 4$: " $x+y = 4$

$y = 4 - x$

$$\mathcal{L} = \ln(x^2 y) - x - y - \lambda_1(-x-y) - \lambda_2(-x) - \lambda_3(-y)$$

$$\mathcal{L} = \underline{2\ln(x) + \ln(y) - x - y} + \lambda_1(x+y) + \lambda_2 x + \lambda_3 y$$



FOC: $\mathcal{L}'_x = \frac{2}{x} - 1 + \lambda_1 + \lambda_2 = 0$

$\mathcal{L}'_y = \frac{1}{y} - 1 + \lambda_1 + \lambda_3 = 0$

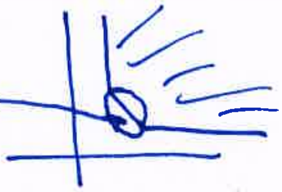
C: $x+y \geq 4$, $x \geq 1$, $y \geq 1$

CSC: $\lambda_1 \geq 0$ and $\lambda_1 \cdot (x+y-4) = 0$

$\lambda_2 \geq 0$ " $\lambda_2 (x-1) = 0$

$\lambda_3 \geq 0$ " $\lambda_3 (y-1) = 0$

$$\underline{x+y=4,\ x>1,\ y>1}$$



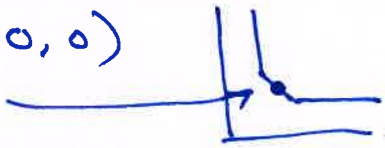
$$\lambda_1 \geq 0 \qquad \lambda_2 = 0 \qquad \lambda_3 = 0$$

FOC: $\begin{cases} \frac{2}{x} - 1 + \lambda_1 + 0 = 0 \\ \frac{1}{y} - 1 + \lambda_1 + 0 = 0 \end{cases}$

$$\lambda_1 = 1 - 2/x$$
$$\lambda_1 = 1 - 1/y$$
$$1 - 2/x = 1 - 1/y$$
$$2/x = 1/y \quad | \cdot xy$$

$$x+y=4 \quad \Leftarrow \quad \underline{2y = x}$$
$$2y + y = 4$$
$$3y = 4$$
$$\underline{y = 4/3} \qquad x = \underline{8/3} \qquad \lambda_1 = 1 - 3/4 = \underline{1/4}$$

$$(x,y;\lambda_1,\lambda_2,\lambda_3) = (8/3, 4/3; 1/4, 0, 0)$$



SOC:

$$h(x,y) = \mathcal{L}(x,y; 1/4, 0, 0)$$
$$= 2\ln x + \ln y - x - y + \tfrac{1}{4}(x+y)$$

$$h'_x = \frac{2}{x} - 1 + 1/4$$
$$h'_y = \frac{1}{y} - 1 + 1/4$$

$$H(h) = \begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$$

$$D_1 = -2/x^2 < 0$$
$$D_2 = 2/(x^2y^2) > 0$$

h concave

$\Downarrow$ SOC

$(x,y) = \underline{(8/3, 4/3)}$ is max

$$f_{max} = f(8/3, 4/3) = 2\ln(8/3) + \ln(4/3) - 8/3 - 4/3$$