Plan

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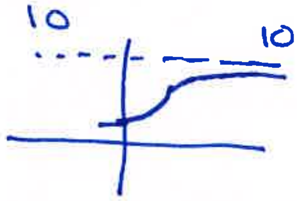
Key Problems

10.2 c) $y' = 5y(1 - y/10) = 5y \cdot \frac{10-y}{10}$



$$\frac{1}{y(10-y)}\, y' = \frac{5}{10}$$

$$\int \frac{1}{y(10-y)}\, y'\, dt = \int \frac{5}{10}\, dt$$

$$\int \frac{A}{y} + \frac{B}{10-y}\, dy = \frac{1}{2}t + C$$

$$\int \frac{1/10}{y} + \frac{1/10}{10-y}\, dy = \frac{1}{2}t + C$$

$$\frac{1}{10}\left(\ln|y| - \ln|10-y|\right) = \frac{1}{2}t + C$$

$$\ln \frac{|y|}{|10-y|} = 5t + 10C$$

$$\frac{|y|}{|10-y|} = e^{5t+C}$$

$$(10-y)\cdot \Big| \quad \frac{y}{10-y} = \pm e^{5t} \cdot e^{C} = K \cdot e^{5t}$$

$$\boxed{\begin{aligned} \frac{1}{y(10-y)} &= \frac{A}{y} + \frac{B}{10-y} \\ 1 &= A\cdot(10-y) + By \\ 1 &= 10A + (B-A)y \\ A = 1/10 \qquad & B - A = 0 \\ & B = 1/10 \end{aligned}}$$

$$y = (10-y)Ke^{5t}$$

$$y + y\cdot Ke^{5t} = 10Ke^{5t}$$

$$\frac{y\cdot(1+Ke^{5t})}{1+Ke^{5t}} = \frac{10Ke^{5t}}{1+Ke^{5t}}$$

$$y = 10 \cdot \frac{Ke^{5t}}{1+Ke^{5t}}$$

4.1 b) $\underbrace{2y-3t^2}_{h'_t}+\underbrace{2(y+t)}_{h'_y}y'=0$ $\quad h'_t+h'_y\cdot y'=0$

① $h'_t=2y-3t^2$

② $h'_y=2y+2t$

$\Rightarrow \boxed{h=2yt-t^3+C(y)}$

$(2yt-t^3+C(y))'_y=2y+2t$

$2t-0+C'(y)=2y+2t$

$C'(y)=2y$

$C(y)=y^2$

$h=2yt-t^3+y^2$

the diff. eqn. is exact.

$\Downarrow$

$2yt-t^3+y^2=C$

$y^2+2t\cdot y+(-t^3-C)=0$

$y=\dfrac{-2t\pm\sqrt{(2t)^2-4\cdot1\cdot(-t^3-C)}}{2}$

$y=\dfrac{-2t\pm\sqrt{4t^2+4t^3+4C}}{2}$

c) $\underbrace{y\dfrac{(1-2\ln t)}{t^3}}_{h'_t}+\underbrace{\dfrac{\ln t}{t^2}}_{h'_y}\cdot y'=0$

① $h'_t=y\cdot\dfrac{1-2\ln t}{t^3}$

② $h'_y=\ln t/t^2$

$h=\dfrac{\ln t}{t^2}\cdot y+C(t)$

$\left(\dfrac{\ln t}{t^2}\cdot y\right)'_t+C(t)=y\cdot\dfrac{1-2\ln t}{t^3}$

$y\cdot\dfrac{1/t\cdot t^2-\ln t\cdot 2t}{t^4}+C'(t)$

$y\cdot\dfrac{t-2t\cdot\ln t}{t^4}+C'(t)=y\cdot\dfrac{1-2\ln t}{t^3}$

$\quad\Vert$

$y\cdot\dfrac{1-2\ln t}{t^3}$

$C'(t)=0$

$C(t)=0$

$h=\dfrac{\ln t}{t^2}\cdot y=C$

exact

$y=\dfrac{Ct^2}{\ln t}$

11.3 c) $y'' - 3y' + 2y = 3e^{2t}$

$\Downarrow$

$y = y_h + y_p = \underline{C_1 e^{2t} + C_2 e^{t} + 3te^{2t}}$

y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = \frac{3 \pm \sqrt{9 - 4\cdot 2}}{2}$

$r_1 = 2, \quad r_2 = 1 \Rightarrow y_h = C_1 e^{2t} + C_2 e^{t}$

y_p: $y'' - 3y' + 2y = 3e^{2t}$

$f(t) = 3e^{2t}$

$f'(t) = 3e^{2t}\cdot 2$

$f''(t) = 6e^{2t}\cdot 2$

$4Ae^{2t} - 3\cdot 2Ae^{2t} + 2\cdot Ae^{2t} = 3e^{2t}$

$(4A - 6A + 2A)e^{2t} = 3e^{2t}$

$0\cdot e^{2t} = 3e^{2t}$

Guess: $y = Ae^{2t}$

$y' = 2Ae^{2t}$

$y'' = 4Ae^{2t}$

$(4At + 4A)e^{2t} - 3(A + 2At)e^{2t} + 2Ate^{2t} = 3e^{2t}$

$(0\cdot At + A)e^{2t} = 3e^{2t}$

$Ae^{2t} = 3e^{2t}$

$A = 3$

$y_p = 3te^{2t}$

Guess: $y = Ate^{2t}$

$y' = A\cdot e^{2t} + At\cdot 2e^{2t} = (A + 2At)e^{2t}$

$y'' = 2Ae^{2t} + (A + 2At)\cdot 2e^{2t} = (4At + 4A)e^{2t}$

Multiply your guess with t

12.2. $y' = Ay$, $A = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix}$ $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Eigenvalues and eigenvectors of A:

$$\begin{vmatrix} -5-\lambda & 0 & 1 \\ 0 & -3-\lambda & 0 \\ 1 & 0 & -5-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda) \cdot \begin{vmatrix} -5-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = 0$$

$\lambda_1 = -3$ $\lambda^2 + 10\lambda + 24 = 0$

$$\lambda = \frac{-10 \pm \sqrt{100 - 4 \cdot 24}}{2} = \frac{-10 \pm 2}{2}$$

$\lambda_2 = -4$, $\lambda_3 = -6$

A is diagonalizable, $\dim E_{-3}, E_{-4}, E_{-6} = 1$

E_{-3}: $\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $x = 2z = 0$, $-3z = 0$ $z = 0$, y free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $E_{-3} = \text{span}(\underline{v_1})$, $\underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

E_{-4}: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ $x = z$, $y = 0$, z free $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

E_{-6}: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $x = -z$, $y = 0$, z free $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

General solution:

$$\underline{y} = C_1 \underline{v_1} \cdot e^{\lambda_1 t} + C_2 \cdot \underline{v_2}\, e^{\lambda_2 t} + C_3 \underline{v_3}\, e^{\lambda_3 t} = C_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-4t} + C_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{-6t}$$

12.3 $\underline{y}_{t+1} = A \cdot \underline{y}_t$, $A = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix}$

(Same as 12.2)

Eigenvectors / eigenvalues: same as in 12.2

General solution:

$$\underline{y}_t = c_1 \cdot \underline{v}_1 \cdot \lambda_1^t + c_2 \cdot \underline{v}_2 \lambda_2^t + c_3 \cdot \underline{v}_3 \lambda_3^t$$

$$= c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot (-3)^t + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot (-4)^t + c_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot (-6)^t$$

4 Exam problems

Final exam 06/2018

first order linear
$y' + a(t)y = b(t)$

2b) $y' + 2t\,y = 4e^{-t^2}$

integrating factor
superposition

Integrating factor:

$a(t) = 2t$
$\int a(t)\,dt = t^2 + C$ } $u = e^{t^2}$ int. factor

$\boxed{u = e^{\int a(t)\,dt}}$

$(y \cdot e^{+t^2})' = 4\cancel{e^{-t^2}} \cdot \cancel{e^{t^2}} = 4$

$y e^{+t^2} = 4t + C$

$y = (4t + C)e^{-t^2} = \underbrace{4te^{-t^2}}_{y_p} + \underbrace{Ce^{-t^2}}_{y_h}$

c) $ty' = y \cdot \ln(y)$

$y' = \dfrac{y \cdot \ln(y)}{t} = \underbrace{\dfrac{1}{t}}_{f(t)} \cdot \underbrace{y \ln(y)}_{g(y)}$ separable

$\dfrac{1}{y \cdot \ln(y)}\, y' = \dfrac{1}{t}$

$\int \dfrac{1}{y \ln y}\, dy = \int \dfrac{1}{t}\, dt$

$\boxed{\begin{array}{l} u = \ln y \\ du = 1/y \cdot dy \end{array}}$

$\int \cancel{\frac{1}{y}} \cdot \frac{1}{u} \cdot \frac{du}{\cancel{(1/y)}}$

$\int \frac{1}{u}\, du = \int \frac{1}{t}\, dt$

$\ln|u| = \ln|t| + C$

$e^{\ln|u|} = e^{\ln|t| + C}$

$|u| = |t| \cdot e^C$

$u = (\pm e^C)\, t = Kt$

$\ln y = Kt$

$y = e^{Kt}$

Final exam 05/2015

$t > 0$

2b) $t^2 y' + ty = \ln t \quad | \cdot 1/t^2$

$$y' + \frac{1}{t}y = \frac{\ln t}{t^2}$$

$\boxed{y' + a(t)y = b(t) \quad \text{linear}}$

Integrating factor:

$$\int \frac{1}{t}\,dt = \ln|t| + C$$

$$u = e^{\ln|t|} = |t| = t$$

$$(y \cdot t)' = \frac{\ln t}{t^2} \cdot t = \frac{\ln t}{t}$$

$$y \cdot t = \int \frac{\ln t}{t}\,dt = \int \cancel{\frac{1}{t}} \cdot u \cdot \frac{du}{\cancel{(1/t)}}$$

$\boxed{\begin{aligned} u &= \ln t \\ du &= (1/t) \cdot dt \end{aligned}}$

$$y \cdot t = \int u\,du$$

$$yt = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln t)^2 + C$$

$$y = \frac{\frac{1}{2}(\ln t)^2 + C}{t} = \frac{(\ln t)^2 + 2C}{2t}$$

Final exam 12/2012

3 a) $y'' - 5y' + 6y = 10e^{-t}$

$$y = y_h + y_p = \underline{C_1 e^{2t} + C_2 e^{3t} + \frac{5}{6} e^{-t}}$$

y_h: $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$

$r_1 = 2,\ r_2 = 3 \rightarrow y_h = \underline{C_1 e^{2t} + C_2 e^{3t}}$

y_p: $y'' - 5y' + 6y = 10e^{-t}$

$f = 10e^{-t}$
$f' = -10e^{-t}$
$f'' = 10e^{-t}$
$\Downarrow$
$y = Ae^{-t}$
$y' = -Ae^{-t}$
$y'' = Ae^{-t}$

$(Ae^{-t}) - 5(-Ae^{-t}) + 6(Ae^{-t}) = 10e^{-t}$

$12A \cdot e^{-t} = 10e^{-t}$

$12A = 10 \quad A = 10/12 = \underline{5/6}$

$y_p = \underline{\frac{5}{6} e^{-t}}$

b) $\underbrace{4te^{2t} y}_{h'_t} \underbrace{- (1-2t)e^{2t} y'}_{h'_y} = 0, \quad t > 1/2$

$h'_t = 4te^{2t} y$

$h'_y = (2t-1)e^{2t}$

$h = \underline{(2t-1)e^{2t} \cdot y + \alpha(t)}$

$h'_t = [(2t-1)e^{2t} y + \alpha(t)]'_t$

$= y \cdot (2e^{2t} + (2t-1)e^{2t} \cdot 2) + \alpha'(t)$

$= \cancel{ye^{2t}(\cancel{2} + 4t - \cancel{2})} + \alpha'(t) \qquad \cancel{4te^{2t} y}$

$\alpha'(t) = 0 \Rightarrow \alpha(t) = 0$

exact with $h = (2t-1)e^{2t}$ $y = C$

$$y = \frac{C}{(2t-1)e^{2t}}$$

(2) Workbook

12.7 $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$ — Second order linear difference eqn.

$$x_t = x_t^h + x_t^p$$
$$= C_1 \cdot (-1)^t + C_2 t (-1)^t + 2^t$$

x_t^h: $x_{t+2} + 2x_{t+1} + x_t = 0$ $x_t = r^t$

$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$
$$r_1 = r_2 = -1$$

$x_t^h = C_1 \cdot (-1)^t + C_2 \cdot t \cdot (-1)^t$

x_t^p: $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$

$$(4A \cdot 2^t) + 2(2A \cdot 2^t) + (A \cdot 2^t) = 9 \cdot 2^t$$
$$(9A) \cdot 2^t = 9 \cdot 2^t$$
$$9A = 9$$
$$A = 1$$

$$x_t^p = 2^t$$

$f_t = 9 \cdot 2^t$

$f_{t+1} = 9 \cdot 2^{t+1} = 9 \cdot 2^t \cdot 2 = 18 \cdot 2^t$

$f_{t+2} = 9 \cdot 2^{t+2} = 9 \cdot 2^t \cdot 4 = 36 \cdot 2^t$

$x_t = A \cdot 2^t$

$x_{t+1} = A \cdot 2^{t+1} = A \cdot 2^t \cdot 2 = 2A \cdot 2^t$

$x_{t+2} = A \cdot 2^{t+2} = 4A \cdot 2^t$

(3) Differential Equations

2.4 $\quad y' = Ay, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\left[(4-\lambda)(-\lambda)-4\right] - 2\left[-2\lambda-2\right] + 2\left[\cancel{4}-\cancel{(4}-\lambda)\right] = 0$$

$$(1-\lambda)\cdot(\lambda^2-4\lambda-4) + 4\lambda + 4 + 2\lambda = 0$$

$$\lambda^2 - 4\lambda - \cancel{4} - \lambda^3 + 4\lambda^2 + 4\lambda + 6\lambda + \cancel{4} = 0$$

$$-\lambda^3 + 5\lambda^2 + 6\lambda = 0$$

$$-\lambda(\lambda^2 - 5\lambda - 6) = 0$$

$\underline{\lambda_1 = 0}$ or $\lambda^2 - 5\lambda - 6 = 0$

$$\lambda = \frac{+5 \pm \sqrt{25 + 4\cdot 6}}{2} = \frac{+5 \pm 7}{2}$$

$\underline{\lambda_2 = 6}$ or $\lambda_3 = \underline{-1}$

A diagonalizable

$\underline{\lambda = 0}$: $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{matrix} \\ ]-2 \\ ]-1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ \cancel{0} & \cancel{0} & \cancel{-4} \end{pmatrix}$

$x = -2y$

$x = 2y - 2z$

$-2z = 0 \quad z = 0$

y free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \underline{v_1} = \underline{\underline{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}} \text{ base of } E_0$$

$\underline{\lambda = 6}$: $\begin{pmatrix} -5 & 2 & 2 \\ 2 & -2 & 2 \\ 1 & 2 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -6 \\ 2 & -2 & 2 \\ -5 & 2 & 2 \end{pmatrix} \begin{matrix} \\ ]-2 \\ ]5 \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & -6 \\ 0 & -6 & 14 \\ \cancel{0} & \cancel{12} & \cancel{-28} \end{pmatrix}$

z free $\quad -6y + 14z = 0 \Rightarrow y = \frac{14}{6}z = \frac{7}{3}z$

$$x + 2y - 6z = 0$$

$$x = -2y + 6z = -2\left(\tfrac{7}{3}\cdot z\right) + 6z = \left(\frac{-14}{3} + \frac{18}{3}\right)z = \frac{4z}{3}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4z/3 \\ 7z/3 \\ z \end{pmatrix} = z/3 \cdot \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} \Rightarrow \underline{v_2} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} \text{ base of } E_6$$

$\underline{\lambda = -1:}$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{matrix} -2 \\ -2 \end{matrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ \cancel{0} & \cancel{-2} & \cancel{0} \end{pmatrix}$$

z free $\quad y = 0 \quad x = -2y - z = -z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ base of } E_{-1}$$

<u>General solution:</u>

$$\underline{y} = c_1 \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + c_3 \underline{v_3} e^{\lambda_3 t}$$

$$= c_1 \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} e^{+6t} + c_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$