Solutions: Key problems Lecture 10

1. a) $y' = 3t^2 + 2$

$y = \int 3t^2 + 2\,dt = \underline{t^3 + 2t + C}$

b) $ty' = 1$

$y' = 1/t$

$y = \int 1/t\,dt = \underline{\ln|t| + C}$

c) $y' = t\sqrt{t^2+1}$

$y = \int t\sqrt{t^2+1}\,dt = \int \cancel{t}\sqrt{u}\cdot\frac{du}{2\cancel{t}} = \int \frac{1}{2}u^{1/2}\,du$

$u = t^2+1$
$du = 2t\,dt$

$u \cdot \sqrt{u}$

$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{3/2}\right) + C = \frac{1}{3}u^{3/2} + C = \underline{\underline{\frac{1}{3}(t^2+1)\sqrt{t^2+1} + C}}$

2. a) $y' = 5y$

$\frac{1}{y}y' = 5$

$\int \frac{1}{y}\underbrace{y'\,dt}_{dy} = \int 5\,dt$

$\int \frac{1}{y}\,dy = \int 5\,dt$

$\ln|y| = 5t + C$

have combined C_1, C_2 in the two integrals

$|y| = e^{5t+C} = e^C \cdot e^{5t}$

$y = \pm e^C e^{5t} = Ke^{5t}$

$y = \underline{\underline{Ke^{5t}}}$

b) $y' = y^2 t$

$\frac{1}{y^2}y' = t$

$\int \frac{1}{y^2}y'\,dt = \int t\,dt$

$\int y^{-2}\,dy = \int t\,dt$

$-y^{-1} = \frac{1}{2}t^2 + C$

$-\frac{1}{y} = \frac{1}{2}t^2 + C$

$\frac{1}{y} = -\frac{1}{2}t^2 - C$

$y = \frac{1 \quad (-2)}{-\frac{1}{2}t^2 - C \quad (-2)}$

$= \underline{\underline{\frac{-2}{t^2 + 2C}}}$

c) $y' = 5y(1 - y/10) = 5y \cdot \frac{1}{10} \cdot (10-y)$

$$\frac{1}{y(10-y)} y' = 5 \cdot \frac{1}{10} = \frac{1}{2}$$

$$\int \frac{1}{y(10-y)} y' \, dt = \int \frac{1}{2} \, dt$$

$$\int \frac{1}{y(10-y)} \, dy = \frac{1}{2} t + C$$

$$\int \frac{1/10}{y} + \frac{1/10}{10-y} \, dy = \frac{1}{2} t + C$$

$(10-y)' = -1$

$$\frac{1}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y} \quad \Big| \, y(10-y)$$

$$1 = A \cdot (10-y) + B \cdot y$$
$$= 10A - Ay + By$$
$$1 = 10A + (B-A)y$$

$\Downarrow$ compare coeff.

$B - A = 0$ (no y-term on LHS)

$10A = 1$ (const. term)

$\Downarrow$

$\underline{A = \frac{1}{10}}$ $\quad B = A = \underline{\frac{1}{10}}$

$$\frac{1}{y(10-y)} = \frac{1/10}{y} + \frac{1/10}{10-y}$$

$$\frac{1}{10} \ln|y| - \frac{1}{10} \ln|10-y| = \frac{1}{2} t + C$$

$$\ln|y| - \ln|10-y| = 10\left(\tfrac{1}{2}t + C\right)$$

$$\ln \left| \frac{y}{10-y} \right| = 5t + 10C$$

$$\left| \frac{y}{10-y} \right| = e^{5t+10C}$$

$$\frac{y}{10-y} = \pm e^{10C} e^{5t} = K e^{5t}$$

$$y = K e^{5t} \cdot (10-y)$$

$$y = 10 K e^{5t} - K e^{5t} \cdot y$$

$$y + K e^{5t} y = 10 K e^{5t}$$

$$\underline{\underline{y = \frac{10 K e^{5t}}{1 + K e^{5t}}}}$$

Theory: How to find y_h in a first order linear differential equations with constant coefficients.

D. ff. eqn: $y' + ay = b(t)$ — a: constant, $b(t)$: function

y_h: $y' + ay = 0$

Alt I: Characteristic equation

$r + a = 0 \Rightarrow \underline{r = -a} \Rightarrow y_h = \underline{\underline{C \cdot e^{-at}}}$

$\left\{ \begin{array}{l} y' \mapsto r \\ y \mapsto 1 \end{array} \right\}$

Motivation:

Try $y = e^{rt}$ in $y' + ay = 0$

$\left. \begin{array}{l} y' = e^{rt} \cdot r \\ y = e^{rt} \end{array} \right\}$ $e^{rt} \cdot r + a \cdot e^{rt} = 0$

$e^{rt}(r+a) = 0$

$r + a = 0$

Alt 2: Separable diff. eqn.

$y' + ay = 0$

$y' = -a \cdot y$ separable

$\frac{1}{y} \cdot y' = -a$

$\int \frac{1}{y} dy = \int -a \, dt$

$c_1 + \ln|y| = -at + C_2$

$|y| = e^{-at + C_2 - C_1}$

$\Rightarrow y = \pm e^{-at} \cdot e^{C_2 - C_1}$

$y = \underline{\underline{C e^{-at}}}$

$(C = \pm e^{C_2 - C_1})$

3. a) $y' + 3y = 6$

Alt A: integrating factor $\quad y' + a(t)y = b(t)$

$a(t) = 3 \Rightarrow \int a(t)\,dt = 3t + C \Rightarrow u = \underline{\underline{e^{3t}}}$

$(y' + 3y)e^{3t} = 6e^{3t}$ follows from choice of u (theory)

$(y \cdot e^{3t})' = 6e^{3t}$

$$y e^{3t} = \int 6e^{3t}\,dt = 6 \cdot \frac{1}{3}e^{3t} + C = 2e^{3t} + C$$

$$y = \frac{2e^{3t} + C}{e^{3t}} = 2 + \frac{C}{e^{3t}} = \underline{\underline{2 + C \cdot e^{-3t}}}$$

Alt B: superposition principle

$y = y_h + y_p = \underline{\underline{C \cdot e^{-3t} + 2}}$

$\underline{y_h}$: $y' + 3y = 0$

$r + 3 = 0$

$r = -3 \Rightarrow y_h = C \cdot e^{-3t}$

$\underline{y_p}$: $y' + 3y = 6$ $\quad$ Try: $y = A$ (const.)

$y' = 0$

$\Downarrow$

$y' + 3y = 6$

$0 + 3A = 6$

$\underline{A = 2} \Rightarrow y_p = 2$

b) $y' - 2ty = 4t$ $\qquad u = e^{\int -2t\,dt} = e^{-t^2+C} = \underline{e^{-t^2}}$

$(ye^{-t^2})' = 4te^{-t^2}$

$ye^{-t^2} = \int 4te^{-t^2}dt = \int 4\cancel{t}e^{u}\frac{du}{-2\cancel{t}} = -2\int e^{u}du$

$$\boxed{\begin{array}{c} u=-t^2 \\ du=-2t\,dt \end{array}}$$

$ye^{-t^2} = -2e^{u}+C = -2e^{-t^2}+C$

$y = \dfrac{-2e^{-t^2}+C}{e^{-t^2}} = -2+\dfrac{C}{e^{-t^2}} = \underline{\underline{-2+Ce^{t^2}}}$

c) $y' + 2y = e^t$

Alt A: $u = e^{\int 2dt} = e^{2t}$

$(ye^{2t})' = e^{t}\cdot e^{2t} = e^{3t}$

$ye^{2t} = \int e^{3t}dt = \frac{1}{3}e^{3t}+C$

$y = \frac{1}{3}\frac{e^{3t}}{e^{2t}}+\frac{C}{e^{2t}} = \underline{\underline{\frac{1}{3}e^{t}+Ce^{-2t}}}$

Alt B: $y = y_h + y_p = \underline{\underline{Ce^{-2t}+\frac{1}{3}e^{t}}}$

y_h: $y'+2y=0$
$r+2=0$
$r=-2$ $\Rightarrow y_h = Ce^{-2t}$

y_p: $y'+2y=e^{t}$ $\qquad$ Try: $y=Ae^{t}$ (A const.)

$y' = Ae^{t}$

$y'+2y=e^{t} \Rightarrow Ae^{t}+2(Ae^{t})=e^{t}$

$3Ae^{t}=e^{t}$

$A=1/3$ $\qquad 3A=1 \Rightarrow A=1/3$

$y_p = \frac{1}{3}e^{t}$