

Solutions: Key problems Lecture 2

1. a) $\begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 1 \cdot (4-4) - 2 \cdot (12-2) + 5(6-1) = -20 + 25 = \underline{5}$

b) $\begin{vmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{vmatrix} = 1(1-c^2) - a(a-bc) + b(ac-b) = \underline{1 - a^2 - b^2 - c^2 + 2abc}$

c) $\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 4 & 2 \\ 0 & 2 & 4 \\ 3 & 0 & 0 \end{vmatrix} = 1 \cdot 1 \cdot (16-4) - 3 \cdot 3 \cdot (16-4) = 1 \cdot 12 - 9 \cdot 12 = \underline{-96}$

2. a) $A = \begin{pmatrix} 4 & 1 & 1 & 3 & 7 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 & 0 \end{pmatrix}$
pivot pos.

$M_{123,123} = 1 \cdot (-1) + 3 \cdot 2 = 5 \neq 0 \Rightarrow \text{rk } A = \underline{3}$

b) $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & 7 & 3 \\ 4 & 5 & 11 & 10 \end{pmatrix}$
not pivot pos. pivot pos.

$M_{123,123} = 1 \cdot (-11-35) - 3(22-28) + 2 \cdot (10+4) = -46 + 18 + 28 = 0$

$M_{12,12} = 1 \cdot (-1) - 3 \cdot 2 = -7 \neq 0$

$M_{123,124} = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 4 & 5 & 10 \end{vmatrix} = 1 \cdot (-10-15) - 3 \cdot (20-12) + 4(10+4) = -25 - 24 + 56 = 7 \neq 0$

$\text{rk } A = \underline{3}$

c) $A = \begin{pmatrix} 3 & 0 & 0 & 7 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{pmatrix}$
not pivot pos. not pivot pos.

$M_{123,123} = 3 \cdot 0 = 0$

$M_{12,12} = 3 \cdot 5 = 15 \neq 0$

$M_{123,124} = \begin{vmatrix} 3 & 0 & 7 \\ 0 & 5 & 0 \\ 0 & 5 & 0 \end{vmatrix} = 5 \cdot 0 = 0$

$\text{rk } A = \underline{2}$

3. a) $A = \begin{pmatrix} 1 & 3 & t \\ 2 & 5 & 7 \end{pmatrix}$

$$M_{12,12} = 5 - 6 = -1 \neq 0$$

$\text{rk } A = 2$ for all t

b) $A = \begin{pmatrix} a & 7 & -3 & 5 & 10 \\ 2 & -3 & 1 & 4 & 18 \\ 1 & 24 & -10 & 11 & 12 \end{pmatrix}$

$$M_{123,123} = a(30-24) - 7(-20-1) - 3(48+3) \\ = 6a + 147 - 153 = 6a - 6$$

$a \neq 1: M_{123,123} \neq 0 \Rightarrow \text{rk } A = 3$

$a = 1: M_{123,123} = 0$

$$M_{12,12} = -3 - 14 = -17 \neq 0$$

$$M_{123,124} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & -3 & 4 \\ 1 & 24 & 11 \end{vmatrix} = 1 \cdot (-33 - 96)$$

$$-7(22-4) + 5(48+3) = -129 - 126 + 255 = 0$$

$$M_{123,125} = \begin{vmatrix} 1 & 7 & 10 \\ 2 & -3 & 18 \\ 1 & 24 & 12 \end{vmatrix} = 1(-36 - 432)$$

$$-7(24-18) + 10(48+3) = -468 - 92 + 510 = 0$$

$a=1: A = \begin{pmatrix} 1 & 7 & -3 & 5 & 10 \\ 2 & -3 & 1 & 4 & 18 \\ 1 & 24 & -10 & 11 & 12 \end{pmatrix}$

↑ not pivot pos.
↑ not pivot pos.
↑ not pivot pos.

$\text{rk}(A) = 2$ if $a=1$
 $= 3$ if $a \neq 1$

c) See next page.

4. a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 3 & 9 \end{pmatrix} \begin{vmatrix} 6 \\ 13 \\ 24 \end{vmatrix}$

$$|A| = M_{123,123} = 1(18-3t) - 1(9-t) + 1 \cdot (3-2) \\ = 10 - 2t$$

$t \neq 5: |A| \neq 0 \Rightarrow$ one solution

$t = 5: M_{123,124} = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 13 \\ 1 & 3 & 24 \end{vmatrix}$

$$= 1(48-39) - 1(24-13) + 6(32) \\ = 9 - 11 + 6 = 4 \neq 0$$

$$M_{12,12} = 2 - 1 = 1 \neq 0$$

$t=5: \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 9 \end{pmatrix} \begin{vmatrix} 6 \\ 13 \\ 24 \end{vmatrix}$

one solution if $t \neq 5$
no solutions if $t = 5$

$$b) (A|b) = \left(\begin{array}{cccc|c} 1 & 4 & 5 & -3 & 6 \\ 2 & 3 & 1 & 0 & 4 \\ 1 & 5 & 4 & -8 & 1 \end{array} \right)$$

$$M_{123,123} = 1 \cdot (28 \cdot 5) - 4(8 \cdot 1) + 5(10 \cdot 7) \\ = 23 - 28 + 15 = 10 \neq 0$$

One free variable, inf. many solutions

Std. Gauss: w is free

Possible choices of free variable:

$M_{123,123} \neq 0 \Rightarrow w$ free is possible

$$M_{123,124} = \begin{vmatrix} 1 & 4 & -3 \\ 2 & 3 & 0 \\ 1 & 5 & -8 \end{vmatrix} = -3 \cdot (10 \cdot 7) + 8 \cdot (7 \cdot 8) = -9 + 8 = -1 \neq 0 \Rightarrow z \text{ free is possible}$$

$$M_{123,134} = \begin{vmatrix} 1 & 5 & -3 \\ 2 & 1 & 0 \\ 1 & 4 & 8 \end{vmatrix} = -3 \cdot (8 \cdot 1) + 8(1 \cdot 10) = -27 + 72 = 45 \neq 0 \Rightarrow y \text{ free is possible}$$

$$M_{123,234} = \begin{vmatrix} 4 & 5 & -3 \\ 7 & 1 & 0 \\ 5 & 4 & -8 \end{vmatrix} = -3(28 \cdot 5) - 8(4 \cdot 35) = -69 + 248 \neq 0 \Rightarrow x \text{ free is possible}$$

Any choice of x, y, z, or w as free is possible.

$$3. c) A = \begin{pmatrix} 1 & a & b \\ a & b & 1 \end{pmatrix}$$

$$\text{rk } A = 2, \text{ if } (a,b) \neq (1,1) \\ = 1, \text{ if } (a,b) = (1,1)$$

$$M_{12,12} = b - a^2 \\ M_{12,13} = 1 - ab \\ M_{12,23} = a - b^2$$

$\text{rk } A < 2 \Leftrightarrow$ all 2-minors are zero

$$b - a^2 = 0 \Rightarrow b = a^2 \\ 1 - ab = 0 \\ a - b^2 = 0$$

$$a = b^2 = (a^2)^2 = a^4$$

$$a - a^4 = 0$$

$$a(1 - a^3) = 0$$

$$a = 0 \text{ or } a^3 = 1$$

$$\underline{b = a^2 = 0} \quad a = \sqrt[3]{1} = 1$$

$$a = 1$$

$$\underline{b = a^2 = 1}$$

$$\underline{1 - ab = 0:}$$

$$a = 1, b = 1 : \text{ok}$$

$$a = 0, b = 0 : \text{not poss.}$$

$$\underline{a,b = (1,1):}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\text{rk } A = 1$$

$$\text{rk } A < 2$$

$$\uparrow$$

$$a = b = 1$$