Solutions: Problem Set 3

$$\underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \underline{v_4} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \quad \underline{v_5} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

1.

a)

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 2 & 4 \end{matrix}} \\ \begin{matrix} 1 & -1 \end{matrix} \end{pmatrix}$$

$M_{12,12} = 4 - 2 = 2 \neq 0$

$\Rightarrow \{\underline{v_1}, \underline{v_2}\}$ lin. independent

b)

$$\begin{array}{c} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -2 \\ 1 & -1 & 5 \end{vmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \underline{v_1} & \underline{v_2} & \underline{v_3} \end{matrix} \end{array} = 1 \cdot (20-2) - 1(10+2) + 1(-2-4)$$

$$= 18 - 12 - 6 = \underline{0}$$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ <u>not</u> linearly independent

c)

$$\begin{array}{c} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 3 \end{vmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \underline{v_1} & \underline{v_2} & \underline{v_5} \end{matrix} \end{array} = 1 \cdot (12-1) - 1(6+1) + 1 \cdot (-2-4)$$

$$= 11 - 7 - 6 = -2 \neq 0$$

$\{\underline{v_1}, \underline{v_2}, \underline{v_5}\}$ lin. independent

d) $\begin{vmatrix} 1 & 1 & -1 \\ 4 & -2 & 3 \\ -1 & 5 & -7 \end{vmatrix} = 1\cdot(14-15)-1(-28+3)-1(20-2)$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \underline{v_2} & \underline{v_3} & \underline{v_4} \end{array}$$

$= -1+25-18 = 6 \neq 0$

$\{\underline{v_2}, \underline{v_3}, \underline{v_4}\}$ lin independent

e) $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -2 & 3 \\ 1 & -1 & 5 & -7 \end{pmatrix}$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{v_4} \end{array}$$

is 3×4, rk is at most $\underline{3}$

so

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$ **not** lin. independent

2. a) dim $\underline{V=2}$, Base $\underline{\{\underline{v_1}, \underline{v_2}\}}$ since $\underline{v_1}, \underline{v_2}$ lin. indep.

b) dim $\underline{V=2}$, Base $\underline{\{\underline{v_1}, \underline{v_2}\}}$

since dim $V<3$ and

$\{\underline{v_1}, \underline{v_2}\}$ lin independent

Alt: $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & -2 \\ 1 & -1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -4 \\ 0 & -2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{array}{cc} \uparrow & \uparrow \\ \underline{v_1} & \underline{v_2} \end{array}$$

c) dim $V=\underline{3}$, Base

$\underline{\{\underline{v_1}, \underline{v_2}, v_5\}}$ since $\underline{v_1}, \underline{v_2}, \underline{v_3}$ lin. indep.

d) $\dim V = \underline{3}$ Base $\underline{\{\underline{v_2}, \underline{v_3}, \underline{v_4}\}}$

since $\dim V < 4$ and $\{\underline{v_2}, \underline{v_3}, \underline{v_4}\}$ lin. indep.

<u>Alt</u>:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -2 & 3 \\ 1 & -1 & 5 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \\ 0 & -2 & 4 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ \underline{v_1} & \underline{v_2} & & \underline{v_4} \end{array}$$

Another base is $\underline{\{\underline{v_1}, \underline{v_2}, \underline{v_4}\}}$

<u>3.</u> $A = (\underline{v_1} | \underline{v_2} | \underline{v_3} | \underline{v_4} | \underline{v_5})$

a) <u>Null(A)</u>: $\boxed{A\underline{x} = 0}$ ← solutions of

$$\begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 2 & 4 & -2 & 3 & -1 \\ 1 & -1 & 5 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 5 & -3 \\ 0 & -2 & 4 & -6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 5 & -3 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$ echelon form

$x_1 = -(2x_3 + 4x_5) - x_3 + (-x_5) - x_5 = \underline{-3x_3 - 6x_5}$

$2x_2 = 4x_3 - 5(-x_5) + 3x_5 \Rightarrow x_2 = \underline{2x_3 + 4x_5}$

$-x_4 - x_5 = 0 \Rightarrow x_4 = \underline{-x_5}$

$\boxed{\begin{array}{l} x_3, x_5 : \text{free} \\ x_1, x_2, x_4 : \text{basic} \end{array}}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_3 - 6x_5 \\ 2x_3 + 4x_5 \\ x_3 \\ -x_5 \\ x_5 \end{pmatrix} = x_3 \cdot \underbrace{\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\underline{w}_1} + x_5 \underbrace{\begin{pmatrix} -6 \\ 4 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\underline{w}_2}$$

$$\text{Null}(A) = \underline{\underline{\text{span}(\underline{w}_1, \underline{w}_2)}}$$

Base: $\underline{\underline{\{\underline{w}_1, \underline{w}_2\}}}$ (lin. independent)

b) $\text{rk}\, A = \underline{3} \Rightarrow \dim \text{Col}(A) = \underline{3}$

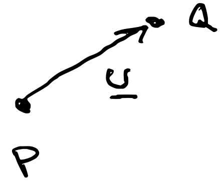
Since there are pivots in col. 1, 2, 4 $\Rightarrow$ Base: $\underline{\underline{\{\underline{v}_1, \underline{v}_2, \underline{v}_4\}}}$

$\text{Col}(A) \subseteq \mathbb{R}^3$ and $\dim \text{Col}(A) = 3$

$\Downarrow$

$$\underline{\underline{\text{Col}(A) = \mathbb{R}^3}}$$

4.



$$P = (1,3,2,5)$$
$$Q = (-2,4,5,1)$$
$$\vec{PQ} = \underline{u} = (-2-1,\ 4-3,\ 5-2,\ 1-5)$$
$$= (-3,1,3,-4)$$

$$(x,y,z,w) = P + t\cdot PQ \qquad (0 \le t \le 1)$$
$$= (1,3,2,5) + t(-3,1,3,-4)$$
$$= \underline{\underline{(1-3t,\ 3+t,\ 2+3t,\ 5-4t)}}$$

or
$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 3 \\ -4 \end{pmatrix}$$

<u>Intersection:</u>
$$w = 9$$
$$5-4t = 9$$
$$-4t = 4 \Rightarrow \underline{t=-1}$$

$$t=-1:\ \underline{\underline{(4,2,-1,9)}}$$

<u>5</u>

$$A = \underbrace{\begin{pmatrix} \cdot & \cdot & \cdots & \cdot \\ \vdots & \ddots & & \vdots \\ \cdot & \cdot & \cdots & \cdot \end{pmatrix}}_{7} 5 \qquad \mathrm{rk}(A) \leq 5$$

$\dim \mathrm{Col}(A) = \mathrm{rk}(A)$ ← # pivots

$\dim \mathrm{Null}(A) = 7 - \mathrm{rk}(A)$ ← # free var's

$$\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A) = \mathrm{rk}(A) + 7 - \mathrm{rk}(A)$$
$$= \underline{\underline{7}}$$