Solutions: Problem Set 4

1.

a) $A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix}$ $\begin{vmatrix} 3-\lambda & 7 \\ 7 & 3-\lambda \end{vmatrix} = 0$

$\lambda^2 - 6\lambda - 40 = 0$

$\lambda_1 = -4, \lambda_2 = 10$

E_{-4}: $\begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix} \to \begin{pmatrix} 7 & 7 \\ 0 & 0 \end{pmatrix}$ $7x + 7y = 0$, y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\underline{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Base of E_{-4}: $\{\underline{u}_1\}$

E_{10}: $\begin{pmatrix} -7 & 7 \\ 7 & -7 \end{pmatrix} \to \begin{pmatrix} -7 & 7 \\ 0 & 0 \end{pmatrix}$ $-7x + 7y = 0$, y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Base of E_{10}: $\{\underline{u}_2\}$

b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda_1 = \lambda_2 = 2$

E_2: $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ $-x + y = 0$, y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Base of E_2: $B = \{\underline{v_1}\}$

c) $A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$ $\begin{vmatrix} 2-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$\lambda^2 - \lambda + 10 = 0$

$\lambda = \frac{1 \pm \sqrt{1 - 4 \cdot 10}}{2}$

no eigenvalues

(among real numbers)

$\Rightarrow$ no eigenvectors

d) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ $\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0$

$(5-\lambda)(\lambda^2 - 8\lambda + 15) = 0$

$\lambda_1 = 5 \quad \lambda_2 = 5, \quad \lambda_3 = 3$

E_5: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ $x = z$, y free, z free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ Base for E_5: $\{\underline{v_1}, \underline{v_2}\}$

E_3: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $x = -z$, $2y = 0$, z free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ Base for E_3: $\{\underline{v_3}\}$

e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(\lambda^2 - 4\lambda + 3) - 1\cdot(2-\lambda-1) + 1\cdot(1-(2-\lambda)) = 0$

$(2-\lambda)(\lambda-3)(\lambda-1) + (\lambda-1) + (\lambda-1) = 0$

$(\lambda-1)[(2-\lambda)(\lambda-3) + 2] = 0$

$\lambda_1 = 1$ or $-\lambda^2 + 5\lambda - 4 = 0$

$\lambda_2 = 1, \quad \lambda_3 = 4$

E_1: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $x+y+z=0$, y, z free $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix}$

$= y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Base for E_1: $\{ \underline{u_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \underline{u_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \}$

E_4: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$ $x = -y+2z = z$, $y = z$, z free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ Base for E_4: $\{ \underline{u_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$

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f) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $\lambda_1 = \lambda_2 = \lambda_3 = 2$

Since A is upper triangular

E_2: $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $y = 0$, $z = 0$, x free $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Base for E_2: $\{ \underline{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$

2.

a) A is diagonalizable with

$$D = \begin{pmatrix} -4 & 0 \\ 0 & 10 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

b) A is not diagonalizable

(dim $E_2 = 1 < 2 \leftarrow$ multiplicity of $\lambda = 2$)

c) A is not diagonalizable

(not enough eigenvalues)

d) A is diagonalizable with

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

e) A is diagonalizable with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

f) A is not diagonalizable

$(\dim E_2 = 1 < 3 \leq \text{mult. of } \lambda = 2)$

3.

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

A symmetric
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A diagonalizable

$$\begin{vmatrix} 1-\lambda & 0 & 0 & -1 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ -1 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\left[(1-\lambda)\cdot(\lambda^2-2\lambda)\right] - (-1)\left[(-1)(\lambda^2-2\lambda)\right] = 0$$

$$(1-\lambda)^2(\lambda^2-2\lambda) - 1\cdot(\lambda^2-2\lambda) = 0$$

$$(\lambda^2-2\lambda)\cdot((1-\lambda)^2-1) = 0$$

$$\lambda(\lambda-2)\cdot(\lambda^2-2\lambda) = 0$$

$$\lambda(\lambda-2)\cdot\lambda\cdot(\lambda-2) = 0$$

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = 2$$