

### Plan

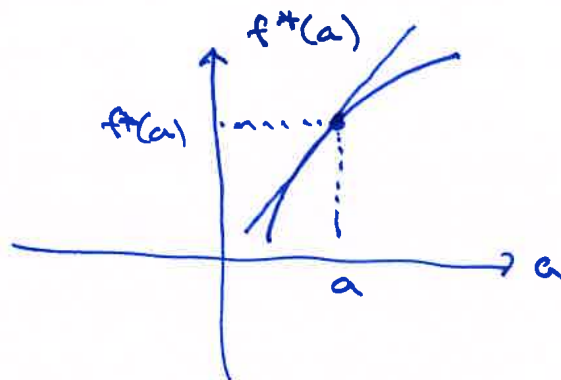
- 1 Introduction to first order differential equations
- 2 Separable differential equations
- 3 Linear first order differential equations
- 4 Exact differential equations

Plenary Session 3 : Monday at 17-20 in A1-040  
 Problems : Lecture 7-9

### Review:

- envelope thm (unconstrained / constrained case)

|                                                                                                                                            |   |                                                                                                                                                                                                                                                                          |
|--------------------------------------------------------------------------------------------------------------------------------------------|---|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\begin{aligned} &\max/\min f(x) \\ &(\text{parameter } a) \\ &\downarrow \\ &\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a)) \end{aligned}$ | } | $\begin{aligned} &\max/\min f(x) \text{ when } \begin{cases} g_1(x) - a_1 = 0 \\ \vdots \\ g_m(x) - a_m = 0 \end{cases} \quad (\leq) \\ &(\text{parameter } a) \\ &\downarrow \\ &\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \underline{\lambda}^*(a)) \end{aligned}$ |
|--------------------------------------------------------------------------------------------------------------------------------------------|---|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|



# ① Introduction to first order differential equations

Defn: A first order diff. eqn. is an equation where the unknown is a function  $y = y(t)$ , which involves the first order derivative  $y' = y'(t)$ .

Ex:  $y' = 2t \Leftrightarrow y'(t) = 2t$

$$y = \int 2t dt = t^2 + C$$

$$y(t) = \underline{\underline{t^2 + C}}$$

general solution  
of the differential eqn.

$C$ : undetermined  
const.

$$y' = 2t, \quad y(0) = 1 \quad \leftarrow \text{initial value problem}$$

$$\Downarrow$$

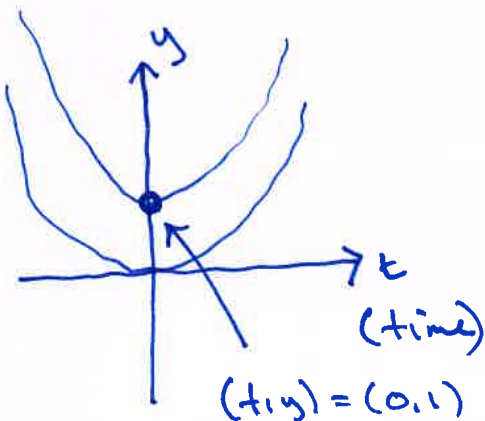
$$y = t^2 + C \quad y(0) = 1: \quad t=0 \Rightarrow y=1$$

$$1 = 0^2 + C$$

$$\underline{\underline{C=1}}$$

$$y = \underline{\underline{t^2 + 1}}$$

particular  
solution



Ex:  $ty' = 2y \Rightarrow y' = \underline{\underline{\frac{2y}{t}}}$

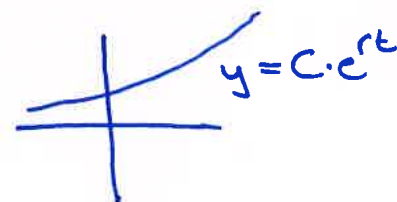
General form of  
"most" first order  
differential equations:

$$y' = F(t, y)$$

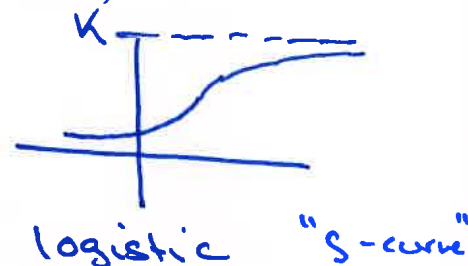
Idea: We use differential equations to model change

For example:

growth rate of  $y$   $\rightarrow$  i)  $y' = r \cdot y$  ( $r$  constant)



ii)  $y' = ry(1 - y/k)$  ( $r, k$  constants)



## ② Separable differential equations

Defn: A first order diff.-eqn. is called separable if it can be written

$$y' = f(t) \cdot g(y)$$

In general:  
 $y' = F(t, y)$

Ex:  $y' = 2y = \underbrace{2}_{f(t)} \cdot \underbrace{y}_{g(y)}$  separable

$$y' + 2y = yt$$

$$y' = yt - 2y$$

$$y' = \underbrace{(t-2)}_{f(t)} \cdot \underbrace{y}_{g(y)}$$

separable

$$y' + y = t$$

$$y' = t - y$$

not separable

Solution method (separable diff. eqn.)

Ex:  $y \cdot y' = t$

$$y' = \frac{t}{y} = \underbrace{t}_{f(t)} \cdot \underbrace{\frac{1}{y}}_{g(y)}$$

separable l.y

$$y y' = t$$

separated form |  $\int - dt$ 

$$\int y y' dt = \int t dt$$

$$\int y dy = \frac{1}{2} t^2 + C$$

$$y' dt = dy:$$

Substitution

$$u = y(t) \\ du = y' \cdot dt$$

$$\frac{1}{2} y^2 + C_1 = \frac{1}{2} t^2 + C_2$$

solution in implicit form

$$\frac{1}{2} y^2 = \frac{1}{2} t^2 + C_2 - C_1 \quad | \cdot 2$$

$$y^2 = t^2 + 2(C_2 - C_1)$$

$$K = 2(C_2 - C_1)$$

$$y = \pm \sqrt{t^2 + 2(C_2 - C_1)}$$

$$\underline{\underline{y = \pm \sqrt{t^2 + K}}}$$

general solution  
in explicit formAlternative:

$$\int y y' dt = \int t dt$$

$$\int y \frac{dy}{dt} \cdot dt = \int t dt$$

$$\int y dy = \int t dt$$

In general:

$$y' = f(t) \cdot g(y)$$

$$| : g(y)$$

$$\frac{1}{g(y)} y' = f(t)$$

$$| \int - dt$$

$$\int \frac{1}{g(y)} y' dt = \int f(t) dt$$

$$y' dt = dy$$

$$\boxed{\int \frac{1}{g(y)} dy = \int f(t) dt}$$

↓ solve both integrals

Implicit solution

↓ solve for y

Explicit general solution

Ex:

$$y' = ry$$

(r constant)

$$| : y$$

$$\frac{1}{y} y' = r$$

$$| \int - dt$$

$$\int \frac{1}{y} y' dt = \int r dt$$

$$\int \frac{1}{y} dy = \int r dt$$

$$\boxed{\ln |y| + C_1 = rt + C_2}$$

$$\ln |y| = rt + \underbrace{C_2 - C_1}_{e'}$$

$$e^{\ln |y|} = e^{rt + C_2 - C_1}$$

$$|y| = e^{rt} \cdot e^{C_2 - C_1}$$

$$y = \boxed{\pm e^{C_2 - C_1}} \cdot e^{rt}$$

$$\underline{y = Ke^{rt}}$$

general  
solution  
(explicit)

## ② Linear first order differential equations

Defn. A first order differential equation is linear if it can be written

$$\boxed{y' + a(t)y = b(t)} \iff y' = \underbrace{b(t) - a(t)y}_{\text{linear in } y}$$

Ex.  $y' + 2y = t^2$   $\begin{cases} a(t) = 2 \\ b(t) = t^2 \end{cases}$  linear

$y' - y = t$   $\begin{cases} a(t) = -1 \\ b(t) = t \end{cases}$  linear

$y' + 3y = 2$   $\begin{cases} a(t) = 3 \\ b(t) = 2 \end{cases}$  linear, const. coeff.

Solution method: Integrating factor (works for all linear first order diff. eqn.)

Ex:  $y' - y = t$   $\cdot u(t)$   $\leftarrow u(t)$ : integrating factor

$$u \cdot y' - u y = u \cdot t \quad | \int - dt$$

$$\int (u y' - u y) dt = \int u(t) t dt$$

$$\int (u y)' dt = \int t u(t) dt$$

$$u y = \int t u(t) dt$$

$$y = \frac{1}{u(t)} \cdot \int t \cdot u(t) dt$$

We want:

$$u y' - u y = (u y)'$$

$$(u y)' - u y = (u y)' + u' y$$

$\equiv$

$$-u = u'$$

$$u' = -1 \cdot u$$

$$u = K e^{-t}$$

Conclusion: We choose  $u = e^{-t}$  as integrating factor.

$$y' - y = t \quad | \cdot e^{-t}$$

$$e^{-t} y' - e^{-t} \cdot y = t e^{-t}$$

$$(e^{-t} \cdot y)' = t e^{-t}$$

$$e^{-t} \cdot y = \int t e^{-t} dt$$

$$y = e^t \cdot \int t e^{-t} dt$$

$$y = e^t (-t e^{-t} - e^{-t} + C)$$

$$y = \underline{\underline{-t - 1 + C e^t}}$$

general solution

$$(e^{-t} \cdot y)' = \underline{-e^{-t} \cdot y} + \underline{e^{-t} \cdot y'}$$

$$\begin{aligned} \int t \cdot e^{-t} dt &= \leftarrow \text{int. by parts} \\ &= -t e^{-t} - \int -e^{-t} \cdot 1 dt \\ &= -t e^{-t} + \int e^{-t} dt \\ &= \underline{\underline{-t e^{-t} - e^{-t} + C}} \end{aligned}$$

$\int u' v dt = u v - \int u v' dt$   
 $u = -e^{-t} \quad v = t$   
 $u' = e^{-t} \quad v' = 1$

In general:

$$y' + \underline{a(t)} \cdot y = b(t)$$

$$u y' + u \cdot a(t) y = b(t) u(t)$$

$$(u y)' = b(t) u(t) \quad | \int - dt$$

$$u y = \int b(t) u(t) dt$$

$$y = \underline{\underline{\frac{1}{u(t)} \cdot \int b(t) u(t) dt}}$$

Integrating factor:

$$u(t) = e^{\int a(t) dt}$$

Ex:  $ty' = 2y \Rightarrow y' = \frac{2y}{t} = \frac{2}{t} \cdot y$  separable

$y' - \frac{2}{t}y = 0$  linear  
 $a(t) = -\frac{2}{t}$   
 $b(t) = 0$

Integrating factor:

$a(t) = -\frac{2}{t} \Rightarrow \int a(t) dt = \int -\frac{2}{t} dt$   
 $= -2 \int \frac{1}{t} dt = -2 \ln|t| + C$

$u = e^{\int a(t) dt} = e^{-2 \ln|t|}$

$= e^{\ln|t| \cdot (-2)} = (e^{\ln|t|})^{-2}$

$= |t|^{-2} = \frac{1}{|t|^2} = \frac{1}{t^2}$

$u = \frac{1}{t^2}$

$y' - \frac{2}{t}y = 0 \cdot \frac{1}{t^2}$

$\frac{1}{t^2}y' - \frac{2}{t^3}y = 0$

$(\frac{1}{t^2} \cdot y)' = 0$

$\frac{1}{t^2}y = C$

$y = Ct^2$

general solution

### ③ Exact differential equations

Defn: A first order differential equation is exact if it can be written

$$p(t,y) + q(t,y) \cdot y' = 0$$

where  $\left\{ \begin{array}{l} p(t,y) = h'_t \\ q(t,y) = h'_y \end{array} \right\}$  for some function  $h(t,y)$  in two variables.

Ex:  $2t + 2y \cdot y' = 0$   $\left\{ \begin{array}{l} h'_t = 2t \\ h'_y = 2y \end{array} \right\}$   $h = t^2 + y^2$   
exact form  du, it is exact

Solution method:  $p(t,y) + q(t,y) \cdot y' = 0$   
 $h'_t + h'_y \cdot y' = 0$   
 $\parallel$   
 $\boxed{h(t,y) = C}$

Ex:  $2t + 2y \cdot y' = 0$   
 $\underline{h'_t + h'_y \cdot y' = 0}$

$$h = t^2 + y^2$$

$$h = t^2 + y^2 = C$$

$$y^2 = C - t^2$$

$$y = \pm \sqrt{C - t^2}$$

implicit  
solution

general  
solution  
expl. form

total derivative:  $\frac{dh}{dt} = \frac{d}{dt} (t^2 + y^2)$   
 $= 2t + 2y \cdot y'$   
 $\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt}$   
 $= h'_t + h'_y \cdot y' = 0$

$$\frac{dh}{dt} = 0 \Rightarrow h(t,y) = C$$

Ex:  $(3x^2 - 3y) + (3y^2 - 3x)y' = 0$   $p + q \cdot y' = 0$

Exact form? Yes

$$\left. \begin{aligned} h'_x &= 3x^2 - 3y = p \\ h'_y &= 3y^2 - 3x = q \end{aligned} \right\}$$

looking for  $h$  that is a solution of these two equations

Test:

$p'_y = q'_x \Rightarrow$  there is an  $h$  s.t.  $h'_x = p$ ,  $h'_y = q$   
 $h''_{xy} = h''_{yx}$

$$\left. \begin{aligned} p'_y &= (3x^2 - 3y)'_y = -3 \\ q'_x &= (3y^2 - 3x)'_x = -3 \end{aligned} \right\} \text{there exists an } h \text{ (it is an exact form)}$$

Find  $h$ :

$$\boxed{\begin{aligned} h'_x &= 3x^2 - 3y \\ h'_y &= 3y^2 - 3x \end{aligned}}$$

$$\Rightarrow (1) \quad h'_x = 3x^2 - 3y + 0$$

$$h = \int (3x^2 - 3y) dx$$

$$h = \underline{x^3 - 3xy + Q(y) = y^3}$$

$$(2) \quad h'_y = 3y^2 - 3x$$

$$(x^3 - 3xy + Q(y))'_y = 3y^2 - 3x$$

$$\cancel{x^3} - 3x + Q'(y) = 3y^2 - 3x$$

$$Q'(y) = 3y^2$$

$$Q(y) = y^3$$

$$h = \underline{x^3 - 3xy + y^3} = C$$

$$\underline{\underline{y^3 - 3xy + x^3 = C}}$$

Solution in implicit form

Ex:  $y' = 5y(1 - y/10)$

(separable)  $= \underbrace{5}_{f(t)} \cdot \underbrace{y(1 - y/10)}_{g(y)}$

$$\frac{1}{y(1 - y/10)} y' = 5$$

$$\int \frac{1}{y(1 - y/10)} y' dt = \int 5 dt$$

$$\int \frac{10}{y(10 - y)} dy = 5t + C$$

$$\int \frac{1}{y} + \frac{1}{10 - y} dy = 5t + C$$

$$\ln |y| - \ln |10 - y| = 5t + C$$

$$\ln \frac{|y|}{|10 - y|} = 5t + C \quad |e^{\phantom{x}}$$

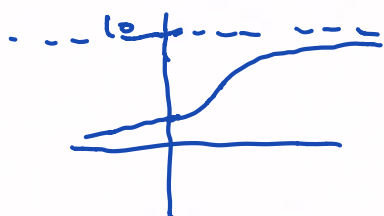
$$\left| \frac{y}{10 - y} \right| = \frac{|y|}{|10 - y|} = e^{5t + C} = e^{5t} \cdot e^C$$

$$\frac{y}{10 - y} = \pm e^C e^{5t} = K \cdot e^{5t} \quad (10 - y)$$

$$y = (10 - y) \cdot K e^{5t} = 10 K e^{5t} - y \cdot K e^{5t}$$

$$y + y \cdot K e^{5t} = 10 K e^{5t}$$

$$\frac{y \cdot (1 + K e^{5t})}{1 + K e^{5t}} = \frac{10 K e^{5t}}{1 + K e^{5t}}$$



$$y' = ry \cdot (1 - y/k)$$

Partial fractions:

$$\frac{10}{y(10 - y)} = \frac{A}{y} + \frac{B}{10 - y} \quad (y(10 - y))$$

$$10 = A \cdot (10 - y) + B y$$

$$10 = (-A + B)y + (10A)$$

$$\begin{cases} -A + B = 0 & B = 1 \\ 10A = 10 & A = 1 \end{cases}$$

$$\begin{aligned} \ln a + \ln b &= \ln(a \cdot b) \\ \ln a - \ln b &= \ln(a/b) \end{aligned}$$

$$y = \frac{10 K e^{5t}}{1 + K e^{5t}}$$

$$= 10 \cdot \frac{K e^{5t}}{1 + K e^{5t}} \quad \downarrow \text{as } t \rightarrow \infty$$

general  
solution