Plan

1. Equilibrium states and stability
2. Superposition principle
3. Introduction to second order differential equations
4. Linear second order equations

[E] Ch. 9

1 Equilibrium states and stability

Defn: A first order differential equation $y' = F(t, y)$ is called autonomous if $y' = F(y)$ is independent of t.

Ex: $y' = 2y$ $\quad y' = 5y(1 - y/10)$ are autonomous

Defn: An equilibrium state $y = y_e$ is a constant solution to $y' = F(y)$. That is, $F(y_e) = 0$.

Ex: 1) $y' = 2y$ $\quad F(y) = \boxed{2y = 0}$

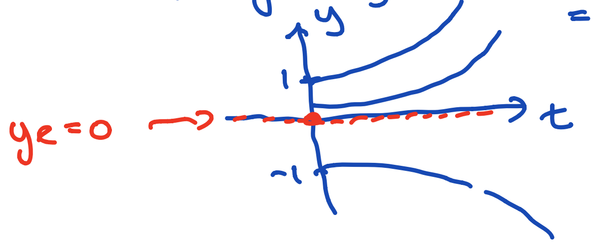
$y = 0 \Rightarrow \underline{y_e = 0}$ eq. state

2) $y' = 5y(1 - y/10)$

$F(y) = 5y(1 - y/10) = 0$

$\underline{y = 0}$ or $1 - y/10 = 0$, $\underline{y = 10}$ $\Rightarrow$ $\underline{y_e = 0}$ and $\underline{y_e = 10}$ eq. states

1) $y' = 2y \Rightarrow y = Ce^{2t} = y_0 \cdot e^{2t}$ $\quad$ $\underline{t = 0}$: $y = C \cdot e^0 = C$, $\quad y_0 = C$

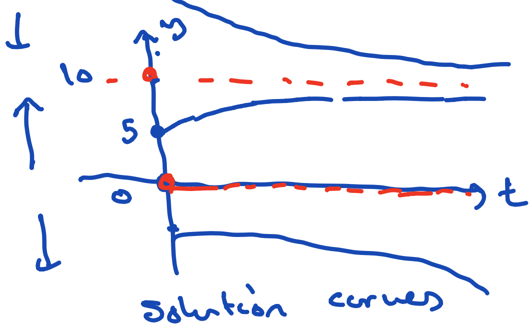


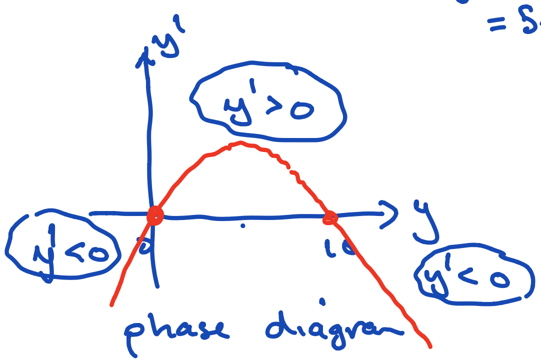
$y_0 = 0$: $y = y_0 \cdot e^{2t} = 0 \cdot e^{2t} = \underline{0}$

2) $y' = 5y(1 - y/10)$

$y_e = 0$ and $y_e = 10$

$y' = 5y(1 - y/10)$
$= 5y - \frac{1}{2}y^2$





$y_e = 10$: stable, not gl. as. stable
$y_e = 0$: unstable

Defn: An equilibrium state $y = y_e$ is called
i) stable if y_0 close to y_e means $y(t) \to y_e$
ii) unstable if y_0 close to y_e " $y(t)$ moves away from y_e
iii) globally asymptotically stable if $y(t) \to y_e$ for all y_0

Stability theorem:
If $y = y_e$ is an equilibrium state of $y' = F(y)$, then
$F'(y_e) > 0 \Rightarrow y_e$ is unstable
$F'(y_e) < 0 \Rightarrow y_e$ is stable

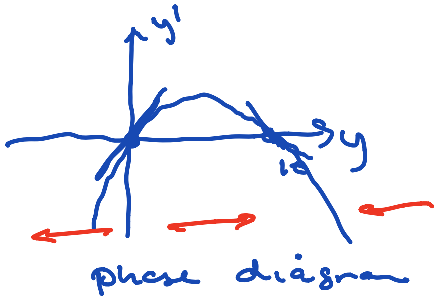
Ex: $y' = 5y(1 - y/10) \Rightarrow y_e = 0$ and $y_e = 10$

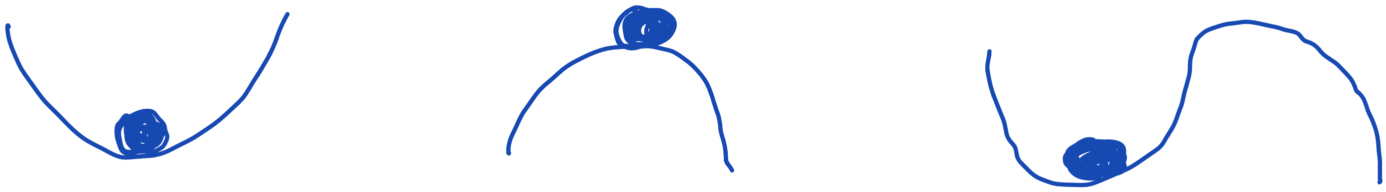
$F(y) = 5y(1 - y/10)$
$= 5y - \frac{1}{2}y^2$

$F'(y) = 5 - y$

$F'(0) = 5 > 0$
$\Downarrow$
$y = 0$ unstable

$F'(10) = -5 < 0$
$y = 10$ stable





stable, gl. as. stable — unstable — stable, not gl. as. stable

(2) Superposition principle

First order linear differential equations: $\boxed{y' + a(t)y = b(t)}$

$D = \frac{d}{dt} + a(t)\cdot$ differential operator

$$y \longrightarrow D(y) = \frac{dy}{dt} + a(t)\cdot y = y' + a(t)\cdot y$$

Ex: $y' + 2y = e^t$ $\qquad D = \left(\frac{d}{dt} + 2\cdot\right)$

$$t^2 \longrightarrow D(t^2) = 2t + 2t^2$$
$$e^t \longrightarrow D(e^t) = e^t + 2e^t = 3e^t$$
$$\tfrac{1}{3}e^t \longrightarrow D(\tfrac{1}{3}e^t) = \tfrac{1}{3}e^t + 2\cdot\tfrac{1}{3}e^t = e^t$$

$\leftarrow$ $y = \frac{1}{3}e^t$ is one solution

D corresponds to a linear first order differential equation $\Longrightarrow$ $D(y_1 + y_2) = D(y_1) + D(y_2)$, $D(c\cdot y_1) = c\cdot D(y_1)$

Superposition principle:

Let $y' + a(t)y = b(t)$ be a linear first order differential equation. Then the general solution is

$$y = y_h + y_p$$

where

y_h: homogeneous solution = general solution of $\boxed{y' + a(t)y = 0}$

y_p: particular solution = one solution of $\boxed{y' + a(t)y = b(t)}$

Why: i) $y' + a(t)y = b(t)$

$D(y) = b(t)$ where $D = \frac{d}{dt} + a(t)$

$y = y_h + y_p \Rightarrow D(y) = D(y_h + y_p)$
$= D(y_h) + D(y_p)$
$= 0 + b(t) = b(t)$

ii) all solutions since

y any solution $\Rightarrow D(y) = b(t)$

Look at $y - y_p$: $D(y - y_p) = D(y) - D(y_p)$
$= b(t) - b(t) = 0$

$\Downarrow$

$y - y_p$ homogeneous solution

$\Downarrow$

$y - y_p = y_h$

$\underline{y = y_h + y_p}$

Ex: $y' - 2y = e^t$ linear $\boxed{\begin{array}{l} a(t) = -2 \\ b(t) = e^t \end{array}}$

$y = y_h + y_p$

In general: $a(t) = a$

$y' + ay = b(t)$

y_h: $y' + ay = 0$
$y' = -ay$
$\Downarrow$
$\underline{y_h = C \cdot e^{-at}}$

y_h: $y' - 2y = 0$ easy to find if $a(t) = a$ is const.

$y' = 2y$

$y = \underline{C \cdot e^{2t}}$

y_p: $y' - 2y = e^t$

$D = \left(\frac{d}{dt} - 2\right)$: $e^t \mapsto D(e^t) = e^t - 2e^t$
$= -e^t$

$-e^t \mapsto D(-e^t) = -e^t - 2(-e^t) = e^t$

$y_p = \underline{-e^t}$

General solution:

$y = y_h + y_p = \underline{\underline{C \cdot e^{2t} - e^t}}$

③ Introduction to second order differential equations

Defn: Diff. equation where the highest order derivative is y''. It can usually be written

$$y'' = F(t, y, y')$$

This is "std. form" for second order diff. eqn.

Ex:

$$y'' = 6t$$

$$y' = \int 6t \, dt = 3t^2 + C$$

$$y' = 3t^2 + C$$

$$y = \int 3t^2 + C \, dt = \underline{t^3 + Ct + D}$$

general solution

two undetermined coefficients

$$y'' = 6t, \quad \underbrace{y(0) = 1, \; y'(0) = 0}_{\text{two initial conditions}}$$

$$y = t^3 + Ct + D \qquad y(0) = 1: \quad 0^3 + C \cdot 0 + D = 1 \qquad \underline{D = 1}$$

$$y' = 3t^2 + C \qquad y'(0) = 0: \quad 3 \cdot 0^2 + C = 0 \qquad \underline{C = 0}$$

$$\Rightarrow \quad y = t^3 + 1 \qquad \text{particular solution}$$

(4) Linear second order differential equations (with constant coefficients)

Defn: A linear second order differential equation can be written

$$y'' + a(t)\cdot y' + b(t)\cdot y = h(t)$$

It is with constant coefficients if $a(t) = a$ and $b(t) = b$:

$$\boxed{y'' + a\cdot y' + by = h(t)}$$

In terms of differential operators:

$$D = \frac{d^2}{dt^2} + a\frac{d}{dt} + b$$

$$D(y) = \left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right)(y) = \underbrace{y'' + ay' + by}_{\text{LHS}}$$

Find all fn. $y = y(t)$ s.t. $D(y) = \underbrace{h(t)}_{\text{RHS}}$

D is linear: $\boxed{\begin{array}{l} D(y_1 + y_2) = D(y_1) + D(y_2) \\ D(c\cdot y_1) = c\cdot D(y_1) \end{array}}$

Ex: $y'' - 7y' + 12y = e^{2t}$

linear second order diff-eqn. with constant coeff.

$$\boxed{\begin{array}{l} a = a(t) = -7 \\ b = b(t) = 12 \end{array}}$$

Super position principle: $\boxed{y = y_h + y_p}$

$D(y) = D(y_h + y_p)$
$= D(y_h) + D(y_p)$
$= 0 + h(t)$
$= h(t)$

The general solution y is $y_h + y_p$, where

y_h: the general solution of the homogeneous eqn. $y'' - 7y' + 12y = 0$

y_p: a particular solution of $y'' - 7y' + 12y = e^{2t}$

Ex: $y'' - 7y' + 12y = e^{2t}$

$y = y_h + y_p = \underbrace{C_1 e^{4t} + C_2 e^{3t}}_{y_h} + \underbrace{\frac{1}{2} e^{2t}}_{y_p} \Rightarrow y = C_1 e^{4t} + C_2 e^{3t} + \frac{1}{2} e^{2t}$

(a) Homogeneous case: y_h

$y'' - 7y' + 12y = 0$

$y'' \to r^2$
$y' \to r$
$y \to 1$

Characteristic equation:

$r^2 - 7r + 12 = 0$

$r = \frac{7 \pm \sqrt{49 - 4 \cdot 12}}{2}$

$= \frac{7 \pm 1}{2} = 4, 3$

$r_1 = 4, \quad r_2 = 3$

$y_h = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$
$= C_1 e^{4t} + C_2 e^{3t} \quad (r_1 \neq r_2)$

$y'' + ay' + by = 0$

Characteristic eqn:

$r^2 + ar + b = 0$

$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

i) $a^2 - 4b > 0$: $r_1 \neq r_2$

$y_h = C_1 \cdot e^{r_1 t} + C_2 e^{r_2 t}$

ii) $a^2 - 4b = 0$: $r_1 = r_2 = -\frac{a}{2}$

$y_h = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$

iii) $a^2 - 4b < 0$: no real solutions

$y_h = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$

$\alpha = -\frac{a}{2}$

$\beta = \frac{\sqrt{4b - a^2}}{2}$

Test:

$y = e^{rt}$
$y' = r \cdot e^{rt}$
$y'' = r e^{rt} \cdot r = r^2 e^{rt}$

does this fit for any values of r?

$y'' - 7y' + 12y =$
$r^2 e^{rt} - 7r e^{rt} + 12 e^{rt}$
$= e^{rt}(r^2 - 7r + 12) = 0$

$r^2 - 7r + 12 = 0$

e^{4t}, e^{3t} are solutions $\Leftarrow$ $r = 4, \; r = 3$

Ex: $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r_1 = r_2 = \underline{3}$

$y = C_1 \underline{e^{3t}} + C_2 \underline{te^{3t}}$

Test: $y = te^{3t}$

$y' = 1 \cdot e^{3t} + t \cdot e^{3t} \cdot 3$
$= (1+3t) e^{3t}$

$y'' = 3e^{3t} + (1+3t) \cdot e^{3t} \cdot 3$
$= (3 + 3 + 9t) e^{3t}$
$= (6 + 9t) e^{3t}$

$y'' - 6y' + 9y$
$= (6 + 9t) e^{3t} - 6 \cdot (1+3t) e^{3t} + 9\, te^{3t}$
$= (\cancel{6} + \cancel{9t} - \cancel{6} - \cancel{18t} + \cancel{9t}) e^{3t}$
$= \underline{0}$

(b) Particular solution : y_p
(inhomogeneous case)

$y'' + ay' + by = h(t)$

Ex: $y'' - 7y' + 12y = e^{2t}$

"Trick": Compute
$h(t) = e^{2t}$
$h'(t) = 2e^{2t}$
$h''(t) = 4e^{2t}$

choose: $y = A \cdot e^{2t}$
$y' = 2Ae^{2t}$
$y'' = 4Ae^{2t}$

check:
$y'' - 7y' + 12y = e^{2t}$
$4Ae^{2t} - 7 \cdot 2Ae^{2t} + 12 \cdot Ae^{2t} = e^{2t}$

Method of undetermined coefficients:

i) Start with $h(t) = e^{2t}$
ii) Make a "guess" based on $h(t)$, but with some undetermined coefficients
iii) Check in the diff.-eqn. and adjust coeff.

$y = e^{rt}$? $y = A \cdot e^{2t}$?

Does $y = Ae^{2t}$ fit for any value of A?

$$e^{2t}(4A - 14A + 12A) = e^{2t} \quad | : e^{2t}$$

$$2A = 1$$

$$A = 1/2 \implies y_p = \frac{1}{2}e^{2t}$$

Ex: $y'' - 2y' + y = t$ second order linear (const. coeff.)

$y = y_h + y_p$ ← superposition

$= C_1e^t + C_2te^t + t + 2$

y_h: $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r_1 = r_2 = 1$

$y_h = C_1e^t + C_2 \cdot te^t$

y_p: $y'' - 2y' + y = t$

$h(t) = t$
$h'(t) = 1$
$h''(t) = 0$

⇓ Guess

$y = At + B$
$y' = A$
$y'' = 0$

check

$0 - 2 \cdot A + (At + B) = t$

$At + (-2A + B) = t$

$A = 1 \qquad -2A + B = 0$

$B = 2A = 2$

$y_p = t + 2$

Ex: $y'' - 3y' + 2y = -e^{2t}$

$$y = y_h + y_p = \underline{\underline{C_1 e^{2t} + C_2 e^{t} - te^{2t}}}$$

y_h: $y'' - 3y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$

$$\underline{r = 2},\ \underline{r = 1} \Rightarrow y_h = \underline{C_1 e^{2t} + C_2 e^{t}}$$

y_p: $y'' - 3y' + 2y = -e^{2t}$

$$h = -e^{2t}$$
$$h' = -2e^{2t}$$
$$\left.h'' = -4e^{2t}\right\} \text{ guess}$$

$$\text{check} \Leftarrow \left\{ \begin{aligned} y &= Ae^{2t} \\ y' &= 2Ae^{2t} \\ y'' &= 4Ae^{2t} \end{aligned} \right.$$

$$(4Ae^{2t}) - 3(2Ae^{2t}) + 2(Ae^{2t}) = -e^{2t}$$

$$e^{2t}(4A - 6A + 2A) = -e^{2t}$$

$$0 \cdot e^{2t} = -e^{2t}$$

try again, multiply with t

$$y = \underline{Ate^{2t}}$$
$$y' = Ae^{2t} + At \cdot 2e^{2t} = \underline{(A + 2At)e^{2t}}$$
$$y'' = 2Ae^{2t} + (A + 2At) \cdot 2e^{2t} = \underline{(4A + 4At)e^{2t}}$$

$\Leftarrow$ check

$$(4A + 4At)e^{2t} - 3(A + 2At)e^{2t} + 2(At)e^{2t} = -e^{2t}$$

$$e^{2t}(4A + \cancel{4At} - 3A - \cancel{6At} + \cancel{2At}) = -e^{2t}$$

$$e^{2t} \cdot A = -e^{2t}$$

$$A = \underline{-1} \Longrightarrow y_p = \underline{-te^{2t}}$$