	Plan
1	Systems of differential equations
2	Introduction to difference equations
3	Systems of difference equations

Review: – equilibrium states and their stability for first order autonomous differential equations: $y' = F(y)$

– second order linear differential equations
$y'' + ay' + by = f(t)$

Superposition principle: $y = y_h + y_p$

(1) Systems of differential equations

Ex: $y_1' = 2y_1 + 5y_2$, $y_2' = 5y_1 + 2y_2$ coupled system

Ex: $\boxed{\begin{aligned} y_1' &= 3y_1 \\ y_2' &= \quad -y_2 \end{aligned}}$ decoupled system $A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$y_1' = 3y_1$
$y_1' - 3y_1 = 0$ $\quad y_1 = y_1^h + y_1^p = c_1 \cdot e^{3t} + 0$ $\Big\}$ $\underline{y_1 = C_1 e^{3t}}$

$\boxed{\text{Char. eqn: } r - 3 = 0,\ r = 3,\ \underset{\shortparallel}{} C e^{3t} \text{ solution}}$

$y_2' = -y_2$ $\quad y_2' + y_2 = 0$ $\quad y_2 = c_2 \cdot e^{-t}$ $\Big\}$ $\underline{y_2 = C_2 e^{-t}}$

Defn: A system of first order differential equations in $y_1, y_2, \ldots, y_n$ has the form

$$y_1' = F_1(t, y_1, y_2, \ldots, y_n)$$
$$y_2' = F_2(t, y_1, y_2, \ldots, y_n)$$
$$\vdots$$
$$y_n' = F_n(t, y_1, y_2, \ldots, y_n)$$

It is called autonomous if $F_1, \ldots, F_n$ are independent of t.

A linear system of first order differential equations has the form

$$\boxed{\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1 \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2 \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n \end{aligned}}$$

where a_{ij}, b_i are constants. It is called homogeneous if $b_1 = b_2 = \cdots = b_n = 0$.

Ex:
$$y_1' = 2y_1 + 5y_2 - 7$$
$$y_2' = 5y_1 + 2y_2 + 3$$

Remark: $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ state vector $\quad \underline{y}' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}$

Matrix notation: $\boxed{\underline{y}' = A \cdot \underline{y} + \underline{b}}$

where A is an $n \times n$-matrix, $\underline{b}$ is an n-vector.

Homogeneous case:

$y' = A \cdot y$

!! decouple the system

Assume that $b = 0$ and that A is diagonalizable: There are n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ of A with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ s.t. $A \cdot v_i = \lambda_i v_i$ for $1 \le i \le n$.

New variables: $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

$\boxed{P^{-1} A P = D}$ when $P = (v_1 | \dots | v_n)$

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$\boxed{y = P \cdot z} \iff z = P^{-1} \cdot y$

$$z' = (P^{-1} \cdot y)' = P^{-1} \cdot y'$$
$$= P^{-1} \cdot (A y) = P^{-1} A y$$
$$= P^{-1} A \cdot (P z) = P^{-1} A P \cdot z = D z$$

$\boxed{z' = D \cdot z}$ decoupled system

$$\begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$\boxed{\begin{aligned} z_1' &= \lambda_1 \cdot z_1 \\ z_2' &= \lambda_2 z_2 \\ &\vdots \\ z_n' &= \lambda_n \cdot z_n \end{aligned}}$$

$z_1' - \lambda_1 z_1 = 0$ lin. homog.

$z_1 = C_1 \cdot e^{r_1 t}$ $r_1 = \lambda_1$

$= C_1 e^{\lambda_1 t}$

General solution:

$$\left.\begin{aligned} z_1 &= C_1 \cdot e^{\lambda_1 t} \\ z_2 &= C_2 \cdot e^{\lambda_2 t} \\ &\vdots \\ z_n &= C_n \cdot e^{\lambda_n t} \end{aligned}\right\} \Rightarrow z = \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix}$$

$$y = P z = (v_1 | v_2 | \dots | v_n) \cdot \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix}$$

$$y = C_1 \cdot v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t} + \dots + C_n v_n e^{\lambda_n t}$$

Summary: Homogeneous case

$\underline{y}' = A\underline{y}$ with A diagonalizable $\Rightarrow$ $\boxed{\text{General solution: } \underline{y} = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} + \dots + C_n \underline{v}_n e^{\lambda_n t}}$

Ex: $y_1' = 2y_1 + 5y_2$, $y_2' = 5y_1 + 2y_2$ $\quad A = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$ symmetric $\Downarrow$ diagonalizable

homogeneous lin system of diff. eqns. $\quad \underline{y}' = A\underline{y}$

i) Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 5 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 21 = 0 \qquad D = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\lambda = \frac{4 \pm \sqrt{16+84}}{2} = 2 \pm 5$$

$\lambda_1 = 7, \quad \lambda_2 = -3$

ii) Eigenvectors:

$\lambda = 7$: $\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \to \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\lambda = -3$: $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \to \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

General solution: $\underline{y} = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t}$

$$= C_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

$$\boxed{\begin{aligned} y_1(t) &= C_1 \cdot e^{7t} - C_2 e^{-3t} \\ y_2(t) &= C_1 e^{7t} + C_2 e^{-3t} \end{aligned}}$$

Connection with second order linear differential equations: Homogeneous case

$$\boxed{y'' + ay' + by = 0} \Rightarrow y'' = -by - ay'$$

We let: $y_1 = y$
$y_2 = y'$

$y_1' = y' = y_2$
$y_2' = y'' = -by - ay' = -by_1 - ay_2$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

i) Second order diff. eqn:

$y = y_h + y_p = y_h$

Char. eqn.: $\boxed{r^2 + ar + b = 0}$

ii) Linear system:

$\mathbf{y}' = A\mathbf{y} \qquad A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix} = 0$$

$-\lambda(-a-\lambda) + b = 0$

$\lambda^2 + a\lambda + b = 0$

Same eqn. !

(2) Difference equations

Defn: A first order *difference equation* is an equation which involves y_{t+1} and y_t, and the unknown is a sequence (y_t).

Ex: $y_{t+1} = 1.05\, y_t$ or $y_{t+1} - 1.1\, y_t = 10$

Sequences: (y_t) : $y_0, y_1, y_2, y_3, \dots$
a numerical value for all non-negative integers $t = 0, 1, 2, \dots$

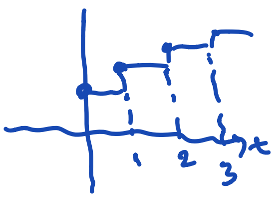
y_t : sequence
$y(t)$: function (cont.)

Ex: $y_0 = 1,\ y_1 = 2,\ y_2 = 4,\ y_3 = 8, \dots$
$y_t = 2^t$, $t \geq 0$ integer

Ex: $y_{t+1} = 1.05\, y_t$
$\underbrace{y_{t+1} - y_t}_{\text{change in the sequence}} = 0.05\, y_t$

General form:
$y_{t+1} = F(t, y_t)$
$\Updownarrow$
$\underbrace{y_{t+1} - y_t}_{\text{change}} = F(t, y_t) - y_t$

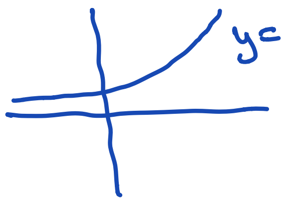
Compare: $\boxed{y_{t+1} = 1.05\, y_t}$
$y_{t+1} - y_t = 0.05\, y_t$



$y_1 = 1.05\, y_0$ $(t=0)$
$y_2 = 1.05\, y_1$ $(t=1)$
$= 1.05(1.05\, y_0) = 1.05^2\, y_0$

$y = C \cdot 1.05^t$, $t = 0, 1, 2, \dots$
$C = y_0$

$y' = 0.05\, y$



$y = C \cdot e^{0.05t}$

The general solution of a first order difference equation will depend on *one* undetermined coeff.

i) Linear first order difference equations:

$$\boxed{y_{t+1} + a \cdot y_t = b_t}$$

a constant
b_t: sequence

← Superposition principle:

$$y_t = y_t^h + y_t^p$$

y_t^h: general solution of the homogeneous eqn. $y_{t+1} + a y_t = 0$

y_t^p: particular solution of $y_{t+1} + a y_t = b_t$

Ex: $y_{t+1} - 1{,}10\, y_t = 10$

$$y_t = y_t^h + y_t^p = \underline{C \cdot 1{,}10^t - 100}$$

y_t^h: $y_{t+1} - 1.10\, y_t = 0$

$r - 1.10 \cdot 1 = 0$

$r = 1.10 \quad \Rightarrow y_t^h = \underline{C \cdot 1{,}10^t}$

$(y_t = C \cdot r^t)$ ← use $C \cdot r^t$ (not $C \cdot e^{rt}$)

Alt: $y_{t+1} = 1{,}10 \cdot y_t$

$y_1 = 1{,}10 \cdot y_0$

$y_2 = 1{,}10\, y_1 = 1{,}10^2 y_0$

$y_3 = 1{,}10 \cdot 1{,}10^2 y_0 = 1{,}10^3 y_0$

$\vdots$

$\Big\}\quad y_t = \underline{C \cdot 1{,}10^t}$

y_t^p: $y_{t+1} - 1{,}10 y_t = 10$

$b_t = 10$
$b_{t+1} = 10$

$\underline{y_t = A}$

$A - 1{,}10 A = 10$

$-0{,}10 A = 10$

$A = \dfrac{10}{-0.10} = -100$

$y_t^p = \underline{-100}$

ii) Linear second order difference equations.

$$\boxed{y_{t+2} + a y_{t+1} + b y_t = f_t}$$

a, b : constants
f_t : expr. in t

Superposition principle: $y_t = y_t^h + y_t^p$

Ex: $y_{t+2} - 7y_{t+1} + 12 y_t = t$

$$y_t = y_t^h + y_t^p = \underbrace{C_1 \cdot 3^t + C_2 \cdot 4^t}_{y_t^h} + \underbrace{\tfrac{1}{6}t + \tfrac{5}{36}}_{y_t^p}$$

y_t^h : $y_{t+2} - 7y_{t+1} + 12y_t = 0$

$$r^2 - 7r + 12 = 0$$
$$r_1 = 3, \quad r_2 = 4$$

$$\Rightarrow y_t^h = C_1 \cdot r_1^t + C_2 \cdot r_2^t = C_1 \cdot 3^t + C_2 \cdot 4^t$$

y_t^p : $y_{t+2} - 7y_{t+1} + 12y_t = t$

$b_t = t$
$b_{t+1} = t+1$
$b_{t+2} = t+2$

$y_t = At + B$
$y_{t+1} = A(t+1) + B = At + (A+B)$
$y_{t+2} = A(t+2) + B = At + (2A+B)$

If $r_1 = r_2$:
$y_t^h = C_1 \cdot r_1^t + C_2 \cdot t \cdot r_1^t$

If there are no real roots:
solutions in terms of sin / cos

$$At + (2A+B) - 7(At + A + B) + 12(At + B) = t$$

$$6At + (-5A + 6B) = t$$

$6A = 1 \qquad A = 1/6$
$-5A + 6B = 0$
$6B = 5A = 5/6 \quad \Big| \, /6$
$B = 5/36$

$$y_t^p = \frac{1}{6}t + \frac{5}{36}$$

Note: $y_{t+1} - y_t$: change $\qquad \Delta y_t = y_{t+1} - y_t$

$\Delta y_{t+1} - \Delta y_t$: second order change

$$\Delta y_{t+1} - \Delta y_t = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) = y_{t+2} - 2y_{t+1} + y_t$$

$$\boxed{y_{t+2} = F(t, y_t, y_{t+1})}$$

$$y_{t+2} - 2y_{t+1} + y_t = F(t, y_t, y_{t+1}) - 2y_{t+1} + y_t$$

3 Systems of difference equations:

Ex:

$$y_{1,t+1} = 0.85\, y_{1,t} + 0.14\, y_{2,t}$$
$$y_{2,t+1} = 0.15\, y_{1,t} + 0.86\, y_{2,t}$$

$$\underline{y}_{t+1} = \begin{pmatrix} 0.85 & 0.14 \\ 0.15 & 0.86 \end{pmatrix} \underline{y}_t = A \cdot \underline{y}_t$$

2x2 system of linear difference equation

In general: If $\underline{y}_{t+1} = A \cdot \underline{y}_t$ and A is a diagonalizable $n\times n$-matrix with linearly independent eigenvector $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ of A with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

$$P = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n) \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

then:

$$\boxed{\underline{y}_t = C_1 \cdot \underline{v}_1 \cdot \lambda_1^t + C_2 \cdot \underline{v}_2 \cdot \lambda_2^t + \dots + C_n \cdot \underline{v}_n \lambda_n^t}$$

is the general solution of the system.

Ex: $\underline{y}_{t+1} = A \cdot \underline{y}_t$ with $A = \begin{pmatrix} 0.85 & 0.14 \\ 0.15 & 0.86 \end{pmatrix}$

$$\begin{vmatrix} 0.85-\lambda & 0.14 \\ 0.15 & 0.86-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1.71\lambda + 0.71 = 0$$

$$\underline{\lambda_1 = 1}, \quad \lambda_2 = \underline{0.71}$$

E_1: $\begin{pmatrix} -0.15 & 0.14 \\ 0.15 & -0.14 \end{pmatrix}$

$$-0.15x + 0.14y = 0$$
$$-15x + 14y$$
$$x = \frac{14}{15} y, \quad y \text{ free}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{14}{15} y \\ y \end{pmatrix} = \frac{y}{15} \begin{pmatrix} 14 \\ 15 \end{pmatrix}$$
$$v_1 = \underline{\begin{pmatrix} 14 \\ 15 \end{pmatrix}}$$

$E_{0.71}$: $\begin{pmatrix} 0.14 & 0.14 \\ 0.15 & 0.15 \end{pmatrix}$

$$14x + 14y = 0$$
$$x = -y, \quad y \text{ free}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

General solution: $y_t = c_1 \cdot \begin{pmatrix} 14 \\ 15 \end{pmatrix} \cdot 1^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 0.71^t$

$$= c_1 \cdot \begin{pmatrix} 14 \\ 15 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 0.71^t$$