	Plan
1	Exam problems: Final exam 11/2019

① $A = \begin{pmatrix} 2 & 1 & 5 & 9 \\ -1 & 1 & 2 & -3 \\ 3 & 0 & 1 & 10 \\ 0 & 3 & 0 & -6 \end{pmatrix}$ $\qquad \underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ col. vectors of A

a) Rank of A:

$$\begin{pmatrix} 2 & 1 & 5 & 9 \\ -1 & 1 & 2 & -3 \\ 3 & 0 & 1 & 10 \\ 0 & 3 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 7 & 6 \\ -1 & 1 & 2 & -3 \\ 3 & 0 & 1 & 10 \\ 0 & 3 & 0 & -6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 7 & 6 \\ 0 & 3 & 9 & 3 \\ 0 & -6 & -20 & -8 \\ 0 & 3 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 7 & 6 \\ 0 & 3 & 9 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -9 & -9 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 7 & 6 \\ 0 & 3 & 9 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{rk } A = \underline{\underline{3}}$$

echelon form

b) $\dim \text{Null}(A) = n - \text{rk}(A) = 4 - 3 = \underline{\underline{1}}$

Null(A): $A\underline{x} = \underline{0}$ $\qquad (A \mid \underline{0}) \rightarrow \cdots \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 7 & 6 & 0 \\ 0 & 3 & 9 & 3 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\quad x \quad y \quad z \quad w$

w free

$-2z - 2w = 0 \Rightarrow z = \underline{-w}$

$3y + 9z + 3w = 0 \Rightarrow 3y = -9z - 3w \qquad y = \underline{2w}$

$\frac{3y}{3} = -9(-w) - 3w = \frac{6w}{3}$

$x + 2y + 7z + 6w = 0$

$\Rightarrow x = -2y - 7z - 6w = -4w + 7w - 6w$

$x = \underline{-3w}$

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3w \\ 2w \\ -w \\ w \end{pmatrix} = w \cdot \begin{pmatrix} -3 \\ 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \underline{w} = \begin{pmatrix} -3 \\ 2 \\ -1 \\ 1 \end{pmatrix} \text{ is a base of Null}(A)$$

c) $\underline{w} = \begin{pmatrix} -3 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, $\quad A\underline{w} = \underline{0}$

$$-3\underline{v}_1 + 2\underline{v}_2 - \underline{v}_3 + \underline{v}_4 = \underline{0}$$

$$\underline{v}_4 = 3\underline{v}_1 - 2\underline{v}_2 + \underline{v}_3$$

② $A = \begin{pmatrix} 4 & 0 & 6 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$

a) $\underline{v}$ eigenvector of A: $\quad A \cdot \underline{v} = \begin{pmatrix} 4 & 0 & 6 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$

$A \cdot \underline{v} = \lambda \cdot \underline{v}$

ok with $\lambda = 0$

$\Downarrow$

$\underline{v}$ is an eigenvector of A with eigenvalue $\lambda = 0$.

b) $\begin{vmatrix} 4-\lambda & 0 & 6 \\ -1 & 3-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$$(4-\lambda) \cdot [(3-\lambda)(2-\lambda) - 0] + 6 \cdot (-1 - 1 \cdot (3-\lambda)) = 0$$

$$(4-\lambda)(3-\lambda)(2-\lambda) + 6(\lambda - 4) = 0$$

$$(4-\lambda) \cdot [(3-\lambda)(2-\lambda) + 6 \cdot (-1)] = 0$$

$$(4-\lambda)(\lambda^2 - 5\lambda) = 0 \qquad \lambda = 4 \text{ or } \lambda = 0 \text{ or } \lambda = 5$$

$$-(\lambda - 4)\lambda(\lambda - 5) = 0$$

Alt: $(\lambda^2 - 7\lambda + 12)(2-\lambda) + 6\lambda - 24 = 0$

$$-\lambda^3 + 9\lambda^2 - 26\lambda + 6\lambda = 0$$

$$-\lambda^3 + 9\lambda^2 - 20\lambda = 0 \qquad -\lambda(\lambda^2 - 9\lambda + 20) = 0$$

$$\lambda = 0 \text{ or } \lambda = 4 \text{ or } \lambda = 5$$

c) Is A diagonalizable? Eigenvalues: $\lambda = 0, 4, 5$ $(n=3)$
enough eigenvalues,
all of mult. 1
$\Downarrow$
A diagonalizable

(3)

a) $y' - 4y = 10e^{-t}$ linear, first order $\Rightarrow$ use superposition

$$y = y_h + y_p = Ce^{4t} - 2e^{-t}$$

y_h: $y' - 4y = 0$
$r - 4 = 0$ $r = 4$
$y_h = C \cdot e^{4t}$

y_p: $y' - 4y = 10e^{-t}$

$y = Ae^{-t}$
$y' = Ae^{-t} \cdot (-1) = -Ae^{-t}$

$-Ae^{-t} - 4Ae^{-t} = 10e^{-t}$
$-5Ae^{-t} = 10e^{-t}$
$A = -2$

$\Rightarrow y_p = -2e^{-t}$

b) $2t + 2ty^2 + (2y + 2yt^2)y' = 0$ exact? Yes ok

$h'_t + h'_y \cdot y' = 0$: "if this works out, then" $\boxed{h = C}$

$$\boxed{\begin{aligned} h'_t &= 2t + 2ty^2 \\ h'_y &= 2y + 2yt^2 \end{aligned}} \Rightarrow h = t^2 + t^2y^2 + \phi(y) = t^2 + t^2y^2 + y^2$$

$\Downarrow$
$h'_y = \cancel{0} + t^2 \cdot 2y + \phi'(y) = 2y + 2yt^2$
$\phi'(y) = 2y$
$\phi(y) = y^2$

Solution:

$h = C$ $t^2 + t^2y^2 + y^2 = C$ implicit solution

$t^2y^2 + y^2 = C - t^2$

$\dfrac{y^2(t^2+1)}{t^2+1} = \dfrac{C - t^2}{t^2+1}$ $y^2 = \dfrac{C - t^2}{1 + t^2}$

$$y = \pm\sqrt{\frac{C - t^2}{1 + t^2}}$$

c) $\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 4 & 0 & 6 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Lin. system of diff-eqn. $\underline{y}' = A \cdot \underline{y}$

If A is diag: $\underline{y} = c_1 \cdot \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} + c_3 \underline{v}_3 e^{\lambda_3 t}$ is the general solution

A is the matrix from Q2:

$\lambda_1 = 0$ $\quad \underline{v}_1 = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$ from Q2a)

$\lambda_2 = 4$ $\quad \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 5$ $\quad \underline{v}_3 = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$

(from Q2)

$\Rightarrow$ $\underline{y} = C_1 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} e^{0 \cdot t} + C_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{4t} + C_3 \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} e^{5t}$

$$\underline{y} = C_1 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{4t} + C_3 \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} e^{5t}$$

$\lambda = 4$: $\begin{pmatrix} 0 & 0 & 6 \\ -1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 6 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & -2 \end{pmatrix} \cdot 1/3$

$6z = 0 \quad z = 0$
y free
$-x - y = 0 \Rightarrow x = -y$

$\begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 5$: $\begin{pmatrix} -1 & 0 & 6 \\ -1 & -2 & 0 \\ 1 & 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 6 \\ 0 & -2 & -6 \\ 0 & 1 & 3 \end{pmatrix} \cdot \frac{1}{2}$

z free
$-2y - 6z = 0 \Rightarrow y = -3z$
$-x + 6z = 0 \quad x = 6z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6z \\ -3z \\ z \end{pmatrix} = z \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$

(4) min $f = x^2+y^2+z^2 - xy + xz - yz$ s.t. $x+y+z=11$

a)
$$\left.\begin{aligned} f'_x &= 2x - y + z \\ f'_y &= 2y - x - z \\ f'_z &= 2z + x - y \end{aligned}\right\} \quad H(f) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \quad (= 2A, \text{ with})$$

$$A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} D_1 &= 2 \\ D_2 &= 3 \\ D_3 &= 2\cdot 3 + 1\cdot(-1) + 1\cdot(-1) = 4 \end{aligned}\right\} \quad \begin{aligned} &D_1, D_2, D_3 > 0 \\ &H(f) \text{ pos. def.} \\ &\Downarrow \\ &\underline{\underline{f \text{ convex}}} \quad (\text{not concave}) \end{aligned}$$

b) $L = x^2+y^2+z^2 - xy + xz - yz - \lambda(x+y+z)$

$$\begin{aligned} \underline{\text{FOC}}: \quad L'_x &= 2x - y + z - \lambda = 0 \\ L'_y &= 2y - x - z - \lambda = 0 \\ L'_z &= 2z + x - y - \lambda = 0 \\ \underline{\text{C}}: \quad & x + y + z = 11 \end{aligned}$$

4×4 lin. sys $\Downarrow$ can use Gauss

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 0 & 11 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 0 & 11 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 3 & -1 & -3 & 0 \\ 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 11 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 11 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 4 & -3 & 0 \\ 0 & 0 & 1 & 2 & 11 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & -11 & -44 \end{array}\right)$$

echelon form

$-11\lambda = -44 \Rightarrow \underline{\lambda = 4}$

$z + 2\lambda = 11 \Rightarrow z = 11 - 2\cdot 4 = \underline{3}$

$y + z - 2\lambda = 0 \qquad y = -3 + 2\cdot 4 = \underline{5}$

$x + y - 2\lambda = 0 \qquad x = -5 + 2\cdot 4 = \underline{3}$

Cond. pt: $(x,y,z;\lambda) = \underline{(3,5,3;4)}$

SOC: $h = f(x,y,z) - 4(x+y+z)$

$H(h) = H(f)$ pos. defn from a) $\Rightarrow h$ convex

$\Rightarrow \underline{(3,5,3) \text{ min pt}}$

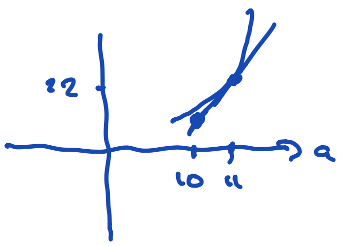
$f_{min} = f(3,5,3) = 3^2 + 5^2 + 3^2 - 3\cdot 5 + 3\cdot 3 - 5\cdot 3 = 43 - 21 = \underline{\underline{22}}$

c) Change c: $x + y + z = \underline{\underline{10}}$

min $f(x,y,z)$ when $x + y + z - a = 0$

$\underline{a = 11}$: $f^*(11) = 22$

$$f^*(10) \cong \underbrace{f^*(11)}_{= 22} + \underbrace{\Delta a}_{= -1} \cdot \underbrace{\frac{df^*(a)}{da}}_{= 4} = \underline{\underline{18}}$$



Env. thm $\dfrac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \lambda^*(a)) = \lambda^*(a) \leftarrow \underline{\lambda^*(11) = 4}$

$L = f - \lambda(x+y+z-a) = f + \lambda(x+y+z) + \lambda a$

⑤ **Extra credit:**

$y_{t+1} = A\cdot y_t$ system of difference eqn., $y_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 4 & 0 & 6 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

A is diagonalizable, have $\lambda_i; \underline{v}_i$ from 3a)

$$y_t = c_1 \cdot \underline{v}_1 \cdot \lambda_1^t + c_2 \underline{v}_2 \lambda_2^t + c_3 v_3 \lambda_3^t = c_1 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} \cdot 0^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot 4^t + c_3 \cdot \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} \cdot 5^t$$

For $t \geq 1$: $0^t = 0$

For $t = 0$: $0^t = 1$

$$y_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$c_1 \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -1 & 6 \\ -1 & 1 & -3 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & -1 & 6 & 1 \\ -1 & 1 & -3 & 1 \\ 2 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & -1 & 7 & 2 \\ -1 & 1 & -3 & 1 \\ 2 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} -1 & -1 & 7 & 2 \\ 0 & 2 & -10 & -1 \\ 0 & -2 & 15 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & -1 & 7 & 2 \\ 0 & 2 & -10 & -1 \\ 0 & 0 & 5 & 4 \end{array}\right)$$

$5z = 4 \quad z = 4/5$

$2y - 10(4/5) = -1$

$\Rightarrow 2y = 8 - 1 = 7$

$y = 7/2$

$-x - y + 7z = 2$

$-x = \frac{7}{2} - 7 \cdot \left(\frac{4}{5}\right) + 2 = \frac{35}{10} - \frac{56}{10} + \frac{20}{10} = \frac{-1}{10}$

$\Rightarrow x = \frac{1}{10}$

$c_1 = \frac{1}{10} \quad c_2 = \frac{7}{2} \quad c_3 = \frac{4}{5}$: $\quad y_t = \frac{1}{10} \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} \cdot 0^t + \frac{7}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} 4^t + \frac{4}{5} \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} \cdot 5^t$

with $0^t = 0$ when $t > 0$

Ex: $y_{t+1} - \lambda \cdot y_t = 0 \Rightarrow y_{t+1} = \lambda y_t \Rightarrow$

$y_1 = \lambda \cdot y_0$

$y_2 = \lambda^2 \cdot y_0$

$y_3 = \lambda^3 \cdot y_0$

$\vdots$

$\boxed{\text{But } y_0 \text{ doesn't have to be } 0!}$

$\lambda = 0$:

$y_1 = 0 \cdot y_0 = 0$

$y_2 = 0^2 \cdot y_0 = 0$

$\vdots$