	Plan
1	Revision: Vectors and matrices (Lecture 1-5)
2	Revision: Optimization problems (Lecture 6-9)
3	Revision: Differential and difference equations (Lecture 10-12)

1 Vectors and matrices Lecture 1-5

Topics:
- linear systems, Gaussian elimination
- vectors, span, linear independence, dimension, base, nullspace, column space
- matrices, determinants / inverse matrix, minors, rank, matrix multiplication, powers
- eigenvalues and eigenvectors, diagonalization, Markov chains, computing A^n
- quadratic forms, definiteness, principal minors, reduced rank criterion, symmetric matrix

Important to know:
(a) Definitions
(b) How to solve problems
(c) Connections / results

Ex: Know defn. of eigenvectors / eigenvalues: $A \cdot \underline{v} = \lambda \cdot \underline{v}$

λ eigenvalue $\Leftrightarrow$ $(A - \lambda I)\underline{v} = \underline{0}$ has non-trivial solu.

$\Leftrightarrow$ $|A - \lambda I| = 0$

i) Rank: A $m \times n$ matrix

$\text{rk}(A) = \#$ pivot positions in A

$=$ max. order of a non-zero minor in A

A $n \times n$: $\boxed{\text{rk} A = n \iff |A| \neq 0}$

ii) Span and linear independence:

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ vectors in $\mathbb{R}^m$

a) $\text{Span}(\underline{v}_1, \dots, \underline{v}_n) =$ all linear combinations

$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \dots + c_n \cdot \underline{v}_n$

b) the vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ are lin. independent if none of the vectors are a lin. comb. of the others

$$\underset{m \times n}{A} = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n)$$

$\text{Col}(A) = \text{span}(\underline{v}_1, \dots, \underline{v}_n)$

$\text{Null}(A) =$ all sol'n of $A \cdot \underline{x} = \underline{0}$

$$\boxed{\begin{array}{l} \dim \text{Col}(A) = \text{rk } A \\ \dim \text{Null}(A) = n - \text{rk } A = \# \text{ free var's} \end{array}}$$

iii) Eigenvalues and eigenvectors:

A $n \times n$ matrix

$A \cdot \underline{v} = \lambda \cdot \underline{v}$

$(A - \lambda I)\underline{v} = \underline{0}$

Solutions:

$\underline{0} \neq \underline{v}$ eigenvector

λ eigenvalue

$$\boxed{\begin{array}{l} \text{a) } \lambda \text{ eigenvalue} \iff \underbrace{|A - \lambda I| = 0}_{\text{ch. eqn.}} \\ \text{b) solve the linear system } (A - \lambda I)\underline{v} = \underline{0} \end{array}}$$

A diagonalizable if $P^{-1}AP = D$ for an invertible matrix P and a diagonal matrix D

it exists if: $\boxed{\begin{array}{l} \text{a) there are } n \text{ eigenval.} \\ \text{b) there are } n \text{ lin. indep. eigenvectors} \end{array}}$

A symmetric $\Rightarrow$ A diagonalizable

iv) <u>Definiteness:</u> A symm. $n \times n$ matrix $\longleftrightarrow$ $f(\underline{x}) = \underline{x}^T \cdot A \cdot \underline{x}$ quadratic form

<u>RRC</u> (reduced rank criterion):

i) A has $\text{rk}\, A = r < n$

$+$

ii) $D_1, D_2, \dots D_r > 0$ (leading principal minors of order up to the rank)

$\Downarrow$

A pos. semidefinite

$f(\underline{x}) \geq 0$ for all $\underline{x} \Longleftrightarrow f$ is pos. semidefin.

$\Updownarrow$

$\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$ (eigenval. of A)

$\Updownarrow$

$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0$ all principal minors of A

<u>Ex:</u> $A = \begin{pmatrix} 4 & 7 & -1 \\ -2 & 0 & 3 \\ 2 & 7 & 2 \end{pmatrix}$

i) $|A| = -(-2) \cdot \begin{vmatrix} 7 & -1 \\ 7 & 2 \end{vmatrix} + (-1) \cdot 3 \cdot \begin{vmatrix} 4 & 7 \\ 2 & 7 \end{vmatrix} = 2(14+7) - 3(28-14)$
$= 2 \cdot 21 - 3 \cdot 14 = 42 - 42 = \underline{0}$

ii) <u>Null(A):</u> $A \cdot \underline{x} = \underline{0}$

$\left(\begin{array}{ccc|c} 4 & 7 & -1 & 0 \\ -2 & 0 & 3 & 0 \\ 2 & 7 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 7 & 2 & 0 \\ -2 & 0 & 3 & 0 \\ 2 & 7 & 2 & 0 \end{array}\right) \begin{matrix} 1 \\ \\ -1 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|c} 2 & 7 & 2 & 0 \\ 0 & 7 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ echelon form

rk A = 2
\# free var = 3 - 2 = 1
n - rk(A)

$2x + 7y + 2z = 0$
$7y + 5z = 0$
z free

$7y = -5z \quad y = -\frac{5}{7} z$

$2x + 7y + 2z = 0$
$2x = -7\left(-\frac{5}{7} z\right) - 2z$
$= 5z - 2z = 3z$
$x = \frac{3}{2} z$

<u>Solutions:</u> $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z/2 \\ -5z/2 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 3/2 \\ -5/2 \\ 1 \end{pmatrix}$
$= \frac{z}{2} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$

<u>Base of Null(A):</u> $\{\underline{w}_1\}$, $\underline{w}_1 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$

dim Null(A) = $\underline{1}$ or $\{\underline{w}_2\}$, $\underline{w}_2 = \begin{pmatrix} 3/2 \\ -5/2 \\ 1 \end{pmatrix}$

iii) Eigenvalues/eigenvectors of A: $A = \begin{pmatrix} 4 & 7 & -1 \\ -2 & 0 & 3 \\ 2 & 7 & 2 \end{pmatrix}$

Eigenvalues:

$|A-\lambda I| = 0$ $\qquad \begin{vmatrix} 4-\lambda & 7 & -1 \\ -2 & -\lambda & 3 \\ 2 & 7 & 2-\lambda \end{vmatrix} = 0$

$\lambda = 0$ is an eigenvalue since $|A| = 0$

$$(4-\lambda)\cdot(-\lambda(2-\lambda)-21) - 7(-2(2-\lambda)-6) - 1(-14+2\lambda) = 0$$
$$(4-\lambda)(\lambda^2-2\lambda-21) - 7(2\lambda-10) - 2(\lambda-7) = 0$$
$$-\lambda^3 + 4\lambda^2 + 2\lambda^2 - 8\lambda + 21\lambda - 84 - 14\lambda + 70 - 2\lambda + 14 = 0$$
$$-\lambda^3 + 6\lambda^2 - 3\lambda = 0$$
$$-\lambda(\lambda^2 - 6\lambda + 3) = 0$$
$$\lambda = 0 \quad \text{or} \quad \lambda^2 - 6\lambda + 3 = 0$$
$$\lambda = \frac{6 \pm \sqrt{36 - 4\cdot 3}}{2} = 3 \pm \frac{\sqrt{24}}{2}$$
$$= 3 \pm \frac{\cancel{\sqrt{4}}\cdot\sqrt{6}}{\cancel{2}} = 3 \pm \sqrt{6}$$
$$\lambda_1 = 0 \qquad \lambda_2 = 3+\sqrt{6} \qquad \lambda_3 = 3-\sqrt{6}$$

Eigenvectors:

E_0: $\lambda = 0$ $\qquad \text{Null}(A - 0\cdot I) = \text{Null}(A) = c\cdot\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$

Base of E_0: $\left\{ \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \right\}$

Ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ $\iff f = x^2 + 2xy + 2y^2 - 2yz + z^2$

symm 3×3 matrix

$D_1 = 1 > 0$

$D_2 = 2 - 1 = 1 > 0$

$|A| = D_3 = -(-1)\cdot(-1-0) + 1\cdot 1$
$= -1 + 1 = 0$

pos. semidefn. or indefn.

Alt 1:
$\Delta_1 = 1, 2, 1 > 0$
$\Delta_2 = 1, 1, 1 > 0$
$\Delta_3 = 0$
all $\Delta_i \geq 0$
$\Downarrow$
A pos. semidefn.

Alt 2:
RRC
$rk A = 2 < 3$
$D_1, D_2 > 0$
$\Downarrow$
A pos. semid.

(2) Optimization problems   Lecture 6–9

- partial derivatives, stationary pts, Hessian, convex/concave fn., second derivative test
- compact sets, extreme value theorem, Lagrange/Kuhn-Tucker problems, FOC+C / FOC+C+CSC, NDCQ, SOC
- envelope theorems

* unconstrained optimization: max/min $f(x_1, \ldots, x_n)$

FOC: $\boxed{f'_{x_1} = \cdots = f'_{x_n} = 0}$ → stationary pts are the sol'n.

f convex ⟺ $H(f)(\underline{x})$ pos. semidefn. for all $\underline{x}$

" concave ⟺ –"– neg. semidefn. –"–

f concave ⇒ any stationary pt. is global max

Env. thm: max/min $f(\underline{x}; a)$

$$\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))$$

classification of all stationary pts: $\boxed{\text{local min, local max, saddle pt.}}$ ← second derivative test

$\underline{x}^*$ stat. pt and $H(f)(\underline{x}^*)$ pos. defn., ⇒ $\underline{x}^*$ local min

* constrained optimization:

Lagrange: max/min $f(\underline{x})$ when $\left\{ \begin{array}{l} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array} \right.$

Kuhn-tucker: max $f(\underline{x})$ when $\left\{ \begin{array}{l} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{array} \right.$
(std. form)

Lagrange: $\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \cdots - \lambda_m \cdot g_m(\underline{x})$

FOC: $\boxed{\begin{array}{l} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{array}}$ C: $\boxed{\begin{array}{l} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array}}$

FOC+C = Lagrange conditions, solution = candidate for max/min

NDCQ: rk J is maximal when $J = \begin{pmatrix} \partial g_1/\partial x_1 & \cdots & \partial g_1/\partial x_n \\ \vdots & & \\ \partial g_m/\partial x_n & \cdots & \partial g_m/\partial x_n \end{pmatrix}$

adm pts that fails NDCQ $\Rightarrow$ exceptional candidates for max/min

i) D (closed, adm pts bounded) $\xRightarrow{\text{EVT}}$ there is a max/min among the candidate pts.

$(\underline{x}^*; \underline{\lambda}^*)$ cand. pt. that satisfy FOC+C:

ii) SOC: $h(\underline{x}) = \mathcal{L}(\underline{x}; \underline{\lambda}^*)$ concave fn. $\Rightarrow$ $\underline{x}^*$ is max

Envelope thm: max/min $f(\underline{x}; a)$ when $\begin{cases} g_1(\underline{x}; a) = 0 \\ \vdots \\ g_m(\underline{x}; a) = 0 \end{cases}$

$$\frac{df^*(a)}{da} = \mathcal{L}'_a(\underline{x}^*(a); \underline{\lambda}^*(a))$$

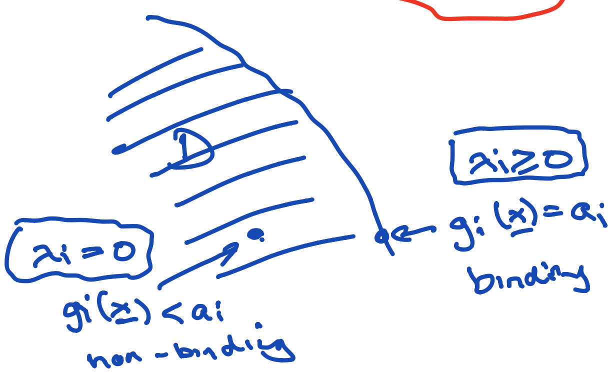
Kuhn-Tucker case: max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

Candidate pts for max: $\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$

FOC: $\boxed{\begin{matrix} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{matrix}}$ C: $\boxed{\begin{matrix} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{matrix}}$ CSC: $\boxed{\begin{matrix} \lambda_1, \dots, \lambda_m \ge 0 \\ \text{and} \\ \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \\ \vdots \\ \lambda_m \cdot (g_m(\underline{x}) - a_m) = 0 \end{matrix}}$

complementary slackness condition

FOC + C + CSC
= Kuhn-Tucker conditions



Ex: max $f = 2x^2 - 4y^2 - 2z^2$ (when $\underbrace{x^4+y^4+z^4}_{g(x,y,z)} \leq \underset{a}{16}$)

a) **Unconstrained pb:** max f

$$\left.\begin{aligned} f'_x &= 4x = 0 \\ f'_y &= -8y = 0 \\ f'_z &= -4z = 0 \end{aligned}\right\} \text{FOC} \qquad \begin{aligned} x&=0\\ y&=0\\ z&=0 \end{aligned}$$

Stat. pts: $(x,y,z) = (0,0,0)$

$$H(f) = \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -4 \end{pmatrix}}_{\text{indefinite}} \qquad \begin{aligned} D_1 &= 4 \\ D_2 &= -32 \end{aligned} \qquad \Rightarrow$$

$(0,0,0)$ saddle pt

no max for f

b) **Kuhn-Tucker:** $\mathcal{L} = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4+y^4+z^4)$

(std form)

$$\boxed{\begin{aligned} &\text{FOC:} && \mathcal{L}'_x = 4x - \lambda \cdot 4x^3 = 0 \\ & && \mathcal{L}'_y = -8y - \lambda \cdot 4y^3 = 0 \\ & && \mathcal{L}'_z = -4z - \lambda \cdot 4z^3 = 0 \\ &\text{C:} && x^4+y^4+z^4 \leq 16 \\ &\text{CSC:} && \lambda \geq 0 \text{ and } (\lambda = 0 \text{ if } x^4+y^4+z^4 < 16) \end{aligned}}$$

FOC:

$$\begin{aligned} 4x\,(1 - \lambda \cdot x^2) &= 0 \\ 4y\,(-2 - \lambda y^2) &= 0 \\ 4z\,(-1 - \lambda z^2) &= 0 \end{aligned} \qquad \boxed{\begin{aligned} x&=0 && \text{or} && \lambda x^2 = 1 \\ y&=0 && \text{or} && \cancel{\lambda y^2 = -2} \\ z&=0 && \text{or} && \cancel{\lambda z^2 = -1} \end{aligned}}$$

$\Downarrow$

$y = z = 0$ and ($x = 0$ or $\lambda x^2 = 1$)

i) $x = y = z = 0$: C: $0 < 16$ ok nonbinding $\Rightarrow \lambda = 0$

$\Rightarrow (0,0,0;0)$ $f = 0$

ii) $y = z = 0,\ \lambda x^2 = 1$: $x^2 = 1/\lambda$ $(\lambda > 0)$ $\lambda > 0 \Rightarrow x^4+y^4+z^4 = 16$

$x^4 + 0 + 0 = 16$

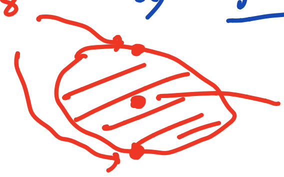
$\Rightarrow (\pm 2, 0, 0; 1/4)$ $f = 8$

$x^4 = 16$

$x^2 = 4 \Rightarrow \lambda = 1/4$

$x = \pm 2$

$f=8$



$(0,0,0;0)$ saddle pt.

Conclusion: $x^4+y^4+z^4 \le 16$ bounded set $\Rightarrow$ EVT there is a max

$-2 \le x \le 2$
$-2 \le y \le 2$
$-2 \le z \le 2$

$f_{max} = 8$

since NDCQ holds at the boundary.

$J = (4x^3 \;\; 4y^3 \;\; 4z^3)$

$\text{rk } J = 0 \Leftrightarrow x=y=z=0$ not on the boundary

(3) Differential and difference equations — Lecture 10-12

i) First order differential equations:

$y' = F(t,y)$

- separable $y' = f(t)\cdot g(y)$
- linear $y' + a(t)y = b(t)$
- exact $p + q\cdot y' = 0$ where $p = h'_t$, $q = h'_y$

Autonomous case: $y' = F(y)$

- equilibrium states, stability

ii) Second order differential equations:

$y'' + ay' + by = f(t)$

- superposition: $y = y_h + y_p$
- y_h: $\boxed{r^2 + ar + b = 0}$

$r_1 \neq r_2$: $y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$r_1 = r_2$: $y_h = C_1 e^{r_1 t} + C_2 t \cdot e^{r_1 t}$

iii) System of linear diff. eqn.

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = A \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

if A diagonalizable

$$y = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + \dots + C_n \cdot \underline{v_n} e^{\lambda_n t}$$

iv) Difference equations:

$y_{t+1} + a y_t = f_t$

$y_{t+2} + a y_{t+1} + b y_t = f_t$

$\Big\}$ first / second order linear diff. eqn.

$\Rightarrow$ superposition

$\underline{y}_{t+1} = A \cdot \underline{y}_t$

system of linear difference eqn.

$\Downarrow$ A diagonalizable

$\underline{y}_t = c_1 \cdot \underline{v}_1 \cdot \lambda_1^t + \dots$

$\dots + c_n \cdot \underline{v}_n \cdot \lambda_n^t$