Plan

1 Definiteness of quadratic forms
2 Non-negative matrices and Markov chains

Review: A $n \times n$ matrix

- eigenvectors and eigenvalues
- diagonalization of A
- how to compute
  - * A^m when m is big
  - * Markov chains

A diagonalizable if
i) there are n eigenvalues
ii) there are n lin. indep. eigenvectors

In that case: $\boxed{P^{-1}AP = D}$

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $P = (\underline{v}_1 | \cdots | \underline{v}_n)$

① Definiteness of quadratic forms

Functions in n variables:

$f(x_1, x_2, \ldots, x_n)$ = some functional expression

$\boxed{\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : f(\underline{x}) = f(x_1, \ldots, x_n)}$

Ex: $f(x_1, x_2, x_3, x_4) = \underbrace{1}_{\text{deg } 0} + \underbrace{x_1 + x_4}_{\text{deg } 1} + \underbrace{x_2 x_3 - x_1 x_4}_{\text{deg } 2} + \underbrace{x_4^3}_{\text{deg } 3}$

poly. deg 3

Defn: A quadratic form in n variables is a polynomial function $f(x_1, \ldots, x_n)$ where each term has degree two.

Ex: $n=1$ $f(x) = ax^2 + bx + c$ quadratic fn. $f(x) = ax^2$ quadratic form

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Ex: $n=2$ $\quad f(x,y) = ax^2 + bxy + cy^2$

x^2, y^2: squares
xy: mixed terms

Ex: $f(x,y) = x^2 + 4xy - y^2$

$f'_x = 2x + 4y = 0$

$f'_y = 4x - 2y = 0$

Notice: For all quadratic forms, we have:

i) $(x,y) = (0,0)$ is a stationary pt ✓

ii) $f(0,0) = 0$ ✓

What kind of stat. pt is this? max? min? Saddle pt?

Matrix form:

Ex: $(x \; y) \cdot \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{x}^T \cdot A \cdot \mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\| $$

$(x+2y \quad 2x-y) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (x+2y)\cdot x + (2x-y)\cdot y$

$= 1x^2 + 2\cdot yx + 2\, x\cdot y + (-1)y^2$

$= x^2 + 4xy - y^2$

$(1,1) \to x^2$ terms
$(1,2),(2,1) \to xy$ terms
$(2,2) \to y^2$ terms

Fact: Any quadratic form $f(x_1,\dots,x_n)$ can be written as

$$f(\mathbf{x}) = \mathbf{x}^T \cdot A \cdot \mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

for a unique symmetric $n \times n$ matrix A.

Ex: $f(x,y,z) = x^2 + \overset{3+3}{6}xy - \overset{-1+(-1)}{2}yz + y^2 - 3z^2$

$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$: $\quad f(\mathbf{x}) = (x \; y \; z) \cdot \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix}$

yz: pos $(2,3)$, $(3,2)$

symmetric matrix of f

Defn: Let $f(x_1, \ldots, x_n)$ be a quadratic form

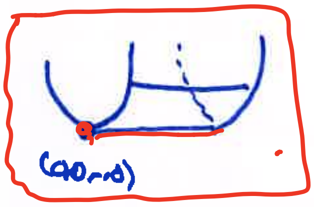
defn = definite

We say that:

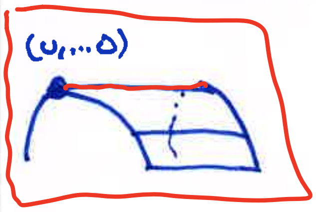
$(0,0,\ldots,0)$

i) f positive defn. if $f(x_1, \ldots, x_n) > 0$ for all $(x_1, \ldots, x_n) \neq (0,0,\ldots,0)$ $\Longrightarrow$ $(0,0,\ldots,0)$ unique global min for f

ii) f negative defn: if $f(x_1, \ldots, x_n) < 0$ for all $(x_1, \ldots, x_n) \neq (0,0,\ldots,0)$ $\Longrightarrow$ $(0,0,\ldots,0)$ unique global max for f



iii) f ~~positive~~ positive semi defn. if $f(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n)$ $\Longrightarrow$ $(0,0,\ldots,0)$ global min for f

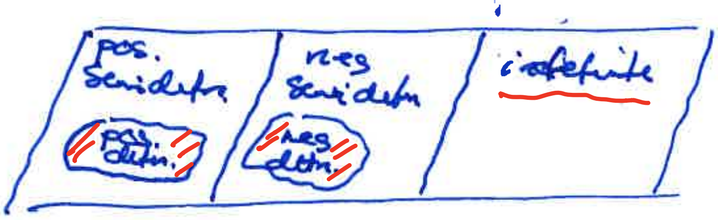


iv) f negative semidefn if $f(x_1, \ldots, x_n) \leq 0$ for all $(x_1, \ldots, x_n)$ $\Longrightarrow$ $(0,0,\ldots,0)$ global max for f

v) f indefinite if $f(x_1, \ldots, x_n)$ can take both positive and negative values $\Longleftrightarrow$ $(0,0,\ldots,0)$ saddle pt. for f

$\Updownarrow$

neither positive nor negative semidefn.



Methods for determining the definiteness (classify)

$f(x_1,\dots,x_n)$ quadratic form $\Longleftrightarrow$ A $n\times n$ symmetric matrix $f(\underline{x}) = \underline{x}^T A \underline{x}$

(A) Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A.

Result: A or f is:

$\lambda_1, \lambda_2, \dots, \lambda_n > 0 \Longleftrightarrow$ positive defn.

$\lambda_1, \lambda_2, \dots, \lambda_n \geq 0 \Longleftrightarrow$ positive semidefn.

$\lambda_1, \lambda_2, \dots, \lambda_n < 0 \Longleftrightarrow$ negative defn.

$\lambda_1, \lambda_2, \dots, \lambda_n \leq 0 \Longleftrightarrow$ negative semidefn.

all other cases (there are pos. and neg. eigenvalues) $\Longleftrightarrow$ indefinite

Ex: $f = x^2 + y^2 + 3z^2 \to A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 3$ pos. defn.

$f(x,y,z) > 0$ when $(x,y,z) \neq (0,0,0)$

$f(x,y,z) = x^2 - y^2 + 3z^2$ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ indefn. $\lambda_1, \lambda_3 > 0$, $\lambda_2 < 0$

$f(1,0,0) = 1 > 0$

$f(0,1,0) = -1 < 0$

$\lambda^2 - 4 = 0$ $\lambda = \pm 2$

$f(x,y) = 4xy$ $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ $\lambda_1 = 2$, $\lambda_2 = -2$ indefn.

$= 4 \cdot \left(\frac{u+v}{\sqrt{2}}\right)\left(\frac{u-v}{\sqrt{2}}\right)$

$= 2(u^2 - v^2) = 2u^2 - 2v^2$

$\lambda_1 = 2$ $\lambda_2 = -2$

$x = \frac{u+v}{\sqrt{2}}$, $y = \frac{u-v}{\sqrt{2}}$

Since A is always symmetric, we can diagonalize; that is

$f(x,y) = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \dots + \lambda_n u_n^2$

where $u_1, \dots, u_n$ are new variables

(B) Principal minors :

Minors where we pick the same rows as columns.

A $n \times n$ symm. matrix

Ex: $A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

2-minors:

9 2-minors $(9 = 3 \cdot 3)$

$$\left\{ \begin{array}{l} M_{12,12} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8 = D_2 = \Delta_2^{12,12} \\ M_{23,23} = \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 2 = \Delta_2^{23,23} \\ M_{13,13} = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3 = \Delta_2^{13,13} \end{array} \right.$$

principal 2-minors of A

D_2 leading principal 2-minor of A

Defn: A principal r-minor of A is an r-minor of the form $M_{I,J}$ with $I = J$.

We write Δ_r for principal r-minors.

A leading principal r-minor of A is

$D_r = M_{\{1,2,\dots,r\},\{1,2,\dots,r\}} = M_{I,I}$ with $I = \{1,2,\dots,r\}$

Result: A $n \times n$ symmetric matrix $\Leftrightarrow$ $f(x_1,\dots,x_n)$ quadr. form

$$\boxed{\begin{array}{ll} D_1, D_2, \dots, D_n > 0 & \Leftrightarrow A \text{ pos. defn.} \\ D_1 < 0, D_2 > 0, D_3 < 0, \dots & \Leftrightarrow A \text{ neg. defn.} \end{array}}$$

$\Updownarrow$

$(-1)^i \cdot D_i > 0$ for $i = 1, 2, \dots, n$.

Ex: $f(x,y,z) = x^2 + 3y^2 + z^2 - xy$ ✓ positive defn

$A = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$D_1 = 1 > 0$ ✓

$D_2 = 3 - (-1/2)^2 = 11/4 > 0$ ✓

$D_3 = +1 \cdot D_2 = 11/4 > 0$ ✓

since $D_1, D_2, D_3 > 0$

Ex: $f = -x^2 - y^2 - 3z^2$ neg. defn.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad \lambda_1 = -1,\ \lambda_2 = -1,\ \lambda_3 = -3$$

negative defn.

$\lambda_1 = D_1 = -1$

$\lambda_1\lambda_2 = D_2 = +1 = (-1)\cdot(-1)$

$\lambda_1\lambda_2\lambda_3 = D_3 = -3 = (-1)(-1)(-3)$

Result: A $n \times n$ symmetric matrix $\leftrightarrow$ $f(x_1, \dots, x_n)$ qu. form

$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0$ for all principal minors $\iff$ positive semi-defn.

$\Delta_1 \leq 0, \Delta_2 \geq 0, \dots$ for all principal minors $\iff$ negative semi-defn

$\Updownarrow$

$(-1)^i \Delta_i \geq 0$ for $i = 1, 2, \dots, n$

Ex: $f = 2x^2 + 8y^2 - 8xy + z^2$ positive semidefn ✓

$$A = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$D_1 = 2$ ✓
$D_2 = 0$ ✓
$D_3 = 0$ ✓

$D_i \geq 0$ for all i

$\Delta_1 = 2, 8, 1 \geq 0$
$\Delta_2 = 0, 8, 2 \geq 0$
$\Delta_3 = 0 \geq 0$

Δ_1: $M_{1,1} = 2$, $M_{2,2} = 8$, $M_{3,3} = 1$

Δ_2: $M_{12,12} = 0$, $M_{23,23} = 8$, $M_{13,13} = 2$

Δ_3: $M_{123,123} = |A| = 0$

(c) Reduced rank criterion

A $n\times n$ symm. $\Rightarrow$ $|A| = 0 = D_n \Leftrightarrow \operatorname{rk} A < n$

A $n\times n$ symm. matrix, $\boxed{r = \operatorname{rk}(A) < n}$

RRC:

$D_1, D_2, \ldots, D_r > 0 \implies A$ pos. semi-defn.

$D_1 < 0, D_2 > 0, \ldots, (-1)^r \cdot D_r > 0 \implies A$ neg. semidefn.

Remember: $\operatorname{rk} A = r < n$ means that $D_{r+1}, D_{r+2}, \ldots, D_n = 0$

Ex: $f(x,y,z) = x^2 + y^2 + z^2 - 2xz$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \operatorname{rk} A = 2 < 3$$

$\operatorname{rk} A = 2$: $r = 2$

$D_1 = 1 > 0$, $D_2 = 1 > 0$ $\implies$ positive semidefn. (RRC)

$(D_3 = 0$ since $\operatorname{rk} A = 2)$

$D_1 = 1$

$D_2 = 1$

$D_3 = +1 \cdot (1-1) = 0$

$\Delta_1 = 1, 1, 1$

$\Delta_2 = 1, 1, 0$

$\Delta_3 = 0$

$\Delta: \geq 0$

$\Downarrow$

Pos. semidefn.

Plan:

1. Definiteness of quadratic forms: More examples
2. Non-negative matrices and Markov chains.

1. Examples: Definiteness

a) $f = x^2 + 4xy + 8xz + 3y^2 - 2yz + 2z^2$

A pos. semidefin. $\Delta_i \geq 0$

A neg — " — $\begin{cases} \Delta_1 \leq 0, \\ \Delta_2 \geq 0, \\ \Delta_3 \leq 0 \end{cases}$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

$D_1 = 1 > 0$
$D_2 = 3 - 4 = -1 < 0$

A indefinite

$$\boxed{\begin{gathered} \Delta_2, \Delta_4, \ldots < 0 \\ \Downarrow \\ \text{indefinite} \end{gathered}}$$

b) $f = 2x^2 + y^2 + 2z^2 + w^2 + 2xz - 2yw$

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$D_1 = 2 > 0$
$D_2 = 2 > 0$
$D_3 = 1 \cdot (4-1) = 3 > 0$
$D_4 = |A| = 0$

RRC:
1) rk $A = 3 < n = 4$
2) $D_1, D_2, D_3 > 0$
$\Downarrow$ RRC
A pos. semidefinite

c) $f = 2xw - 2yz$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} D_1 &= 0 \geq 0 \\ D_2 &= 0 \geq 0 \\ D_3 &= 0 \geq 0 \\ D_4 &= -1 \cdot (-1 \cdot 1) = 1 \geq 0 \end{aligned}}$$

$\Delta_1 = 0, 0, 0, 0 \geq 0$
$\Delta_2 = 0, \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = 0 - 1 = -1 < 0$
$\qquad \qquad \Delta_2^{23,23}$

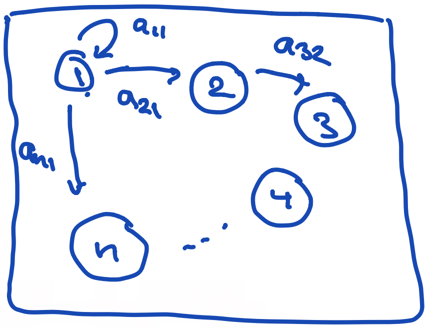
$\Downarrow$
A indefinite

② Non-negative matrices and Markov chains

Defn: $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ $n \times n$ matrix

- A is positive if $a_{ij} > 0$ for all i,j, and write $A > 0$
- A is non-negative if $a_{ij} \geq 0$ for all i,j, and write $A \geq 0$

Markov chains:



State vector:

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ $v_1, \dots, v_n \geq 0$, $v_1 + v_2 + \dots + v_n = 1$

Transition matrix:

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ $a_{ij} \geq 0$ for all i,j

$a_{1j} + a_{2j} + \dots + a_{nj} = 1$ for all j (each column has sum $= 1$)

Dynamic equation:

$$\underline{v}_{t+1} = A \cdot \underline{v}_t$$

Defn: The Markov chain is regular if you can get from any node to any other node in a finite no. of steps.

If $A > 0$ (all $a_{ij} > 0$) then it is regular.

Thm:

Let us consider a regular Markov chain.

Then: ① $\lambda = 1$ is an eigenvalue of A with multiplicity 1; all other eigenvalues have $|\lambda| < 1$

$\lambda = 1$ dominant eigenvalue

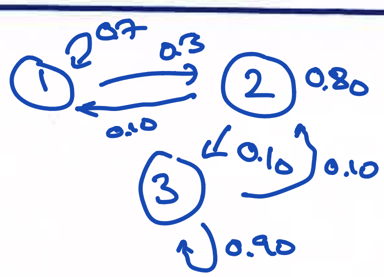
② there is a unique eigenvector $\underline{v}$ of A with eigenvalue $\lambda = 1$ that is a state vector

③ For any starting state $\underline{v}_0$, we have

$$\lim_{t \to \infty} \underbrace{A^t \cdot \underline{v}_0}_{\underline{v}_t} = \underline{v}$$

This $\underline{v}$ is called the equilibrium state.

Example:



regular

$$A = \begin{pmatrix} 0.7 & 0.10 & 0 \\ 0.3 & 0.80 & 0.10 \\ 0 & 0.10 & 0.90 \end{pmatrix}$$

$A^2 > 0$ (no zeros)

Background:

Positive matrices : $A > 0$

Theorem: (Perron)

If $A > 0$, then there is a dominant eigenvalue $\lambda_A > 0$ of multiplicity 1 such that

i) any other eigenvalue λ satisfies $|\lambda| < \lambda_A$

ii) there is a unique vector $\underline{v}_A$ that is an eigenvector of A with eigenvalue λ_A, such that $\underline{v}_A > 0$ and with sum $= 1$.

iii) lowest col. sum $\leq \lambda_A \leq$ highest col. sum

Markov chain with $A > 0$:

special case with $\lambda_A = 1$

Thm (Frobenius)

$A \geq 0$ non-negative matrix

If A is irreducible, then A has a dominant eigenvalue $\lambda_A > 0$ such that

i) $|\lambda| \leq \lambda_A$ for all eigenvalues λ of A

ii) there is a unique state vector $\underline{v}_A$ that is an eigenvector of A with eigenvalue λ_A

iii) lowest col. sum $\leq \lambda_A \leq$ highest col. sum

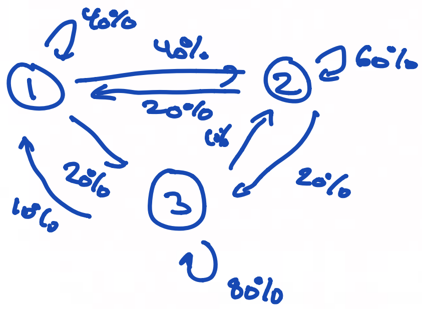
A irreducible means that there is a sequence

$i \to i_1 \to i_2 \to \dots \to i_k \to j$

with $\begin{cases} a_{i i_1} \neq 0 \\ a_{i_1 i_2} \neq 0 \\ \vdots \\ a_{i_k j} \neq 0 \end{cases}$

for any (i,j)

Example: Markov chain



$$A = \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{pmatrix}$$

regular Markov chain

$$\underline{v}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} : \quad \lim_{m \to \infty} A^m \cdot \underline{v}_0$$

Using theory:

$\lambda = 1$: dominant eigenvalue

$$E_1: \begin{pmatrix} -0.6 & 0.2 & 0.1 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.2 \end{pmatrix} \begin{matrix} \cdot 10 \\ \cdot 10 \\ \cdot 10 \end{matrix} \to \begin{pmatrix} 2 & 2 & -2 \\ 4 & -4 & 1 \\ -6 & 2 & 1 \end{pmatrix} \begin{matrix} \\ -2 \\ 3 \end{matrix}$$

$$\to \begin{pmatrix} 2 & 2 & -2 \\ 0 & -8 & 5 \\ \cancel{0} & \cancel{8} & \cancel{-5} \end{pmatrix} \begin{matrix} \\ \\ 1 \end{matrix}$$

$2x + 2y - 2z = 0 \Rightarrow x = -y + z$

$-8y + 5z = 0 \Rightarrow 8y = 5z$

$y = 5z/8$

z free

$x = -5z/8 + z = 3z/8$

$$\underline{v} = \begin{pmatrix} 3z/8 \\ 5z/8 \\ z \end{pmatrix} = \frac{z}{8} \cdot \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$$

Base: $\underline{v}_1 = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$

$$\boxed{\frac{3z}{8} + \frac{5z}{8} + z = 1} \quad | \cdot 8$$

$3z + 5z + 8z = 8$

$16z = 8$

$z = 8/16 = 1/2$

Eq. state:

$$\underline{v} = (3/16, 5/16, 1/2)$$

E_1: $\begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} t$

$3 + 5 + 8 = 16 \Rightarrow t = \frac{1}{16}$

$$\Rightarrow \underline{v} = (3/16, 5/16, 8/16)$$

Remark: Regular Markov chains

The precise definition is as follows:

A Markov chain matrix A is called regular if there is an integer s such that you can get from any node to any other node in exactly s steps.

This means:

$$\boxed{A \text{ regular} \iff A^s \text{ is a positive matrix for some integer } s \geq 1 \text{ (i.e. all entries in } A^s \text{ are positive)}}$$

$A > 0$ positive matrix (all $a_{ij} > 0$) $\Rightarrow$ regular Markov chain

Ex: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

is not regular

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$A^3 = A^2 \cdot A = I \cdot A = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^4 = A^2 \cdot A^2 = I \cdot I = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\vdots$

you can get from any node to any other node, but there is no integer s so that you can get from any node to any other node in exactly s steps ($1 \to 2$, $2 \to 1$: odd no. steps $1 \to 1$, $2 \to 2$: even no. steps)