

Plan

- 1 Constrained optimization: Necessary conditions
- 2 Lagrange problems
- 3 Kuhn-Tucker problems

Note:

- 1) Plenary Session 2
Mon 17-20. (L4-6)
- 2) Mock mid-term exam
next Fri at 15-16.

Review:

— unconstrained optimization

— convex / concave fn.

$$\max/\min f(x_1, x_2, \dots, x_n)$$

$$f \text{ convex} \Rightarrow H(f)(\underline{x})$$

$$f \text{ concave} \Rightarrow H(f)(\underline{x})$$

- i) find stat. pts of f
- ii) classify them loc. (sec. der. test)
- iii) concl. about max/min pos. semi-def. for all $\underline{x}$ neg. — for all $\underline{x}$

f convex: any stat. pt is global min $\cup$

f concave: any stat. pt is global max $\cap$

① Constrained optimization

$\max/\min f(x_1, x_2, \dots, x_n)$ with constraints

one or more equations or inequalities

Ex: $\min f(x, y) = xy$ when $x^2 + y^2 = 10$

$\max f(x, y, z, w) = xw - yz$ when $x + y + z + w \leq 16$

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ w \geq 0 \end{cases}$$

— objective function: $f(\underline{x}) = f(x_1, \dots, x_n)$

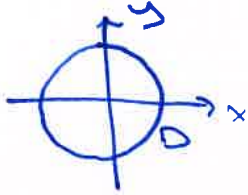
— admissible pts: $D = \{ \underline{x} = (x_1, \dots, x_n) \mid \text{all constraints are satisfied} \}$

The set of all pts in $\mathbb{R}^n$ such that all constraints are satisfied

D subset of $\mathbb{R}^n$
Set of admissible pts

The set D of admissible pts.

Ex: $x^2 + y^2 = 10$



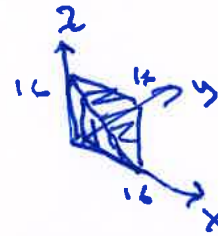
D : circle with
 $r = \sqrt{10}$, center $(0,0)$

$\min f = xy$ when $x^2 + y^2 = 10$

$$x + y + z + w \leq 16$$

$$x \geq 0, y \geq 0, z \geq 0, w \geq 0$$

D in $\mathbb{R}^4$



bounded:

$$0 \leq x \leq 16$$

$$0 \leq y \leq 16$$

$$0 \leq z \leq 16$$

$$0 \leq w \leq 16$$

D : set of admissible pts.

Defn: D is compact if it is closed and bounded

D is closed: $\left\{ \begin{array}{l} \text{when all constraints are} \\ \text{given by } = \text{ (equations)} \\ \text{or } \leq, \geq \text{ (closed inequalities)} \end{array} \right.$

all boundary pts are in D $\rightarrow$

D is bounded: There exists bounds $a_1, \dots, a_n, b_1, \dots, b_n$ (finite numbers) such that

$$a_1 \leq x_1 \leq b_1$$

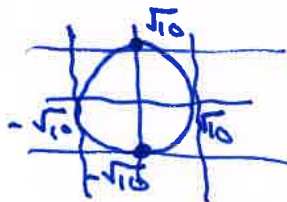
$$a_2 \leq x_2 \leq b_2$$

$\vdots$

$$a_n \leq x_n \leq b_n$$

for all pts $(x_1, x_2, \dots, x_n)$ in D .

Ex: $x^2 + y^2 = 10$



$$b_2 = \sqrt{10}$$

$$a_2 = -\sqrt{10}$$

$$a_1 = -\sqrt{10}$$

$$b_1 = \sqrt{10}$$

Extreme value theorem (EVT)

A continuous fn $f(x_1, \dots, x_n)$ on a compact set D in $\mathbb{R}^n$ has a (global) max and min

② Lagrange problems

max/min $f(x_1, \dots, x_n)$ when

$$\begin{cases} g_1(x_1, \dots, x_n) = a_1 \\ g_2(x_1, \dots, x_n) = a_2 \\ \vdots \\ g_m(x_1, \dots, x_n) = a_m \end{cases}$$

n variables

m constraints (equations)

$a_1, \dots, a_m$: constants

$g_1, \dots, g_m$: any functions

Ex: min $f(x, y) = xy$ when $\overbrace{x^2 + y^2}^{g(x, y)} = \overbrace{10}^a$

$$\begin{aligned} L &= L(x, y; \lambda) = f(x, y) - \lambda \cdot g(x, y) \\ &= xy - \lambda \cdot (x^2 + y^2) \end{aligned}$$

λ : Lagrange multiplier

$$\begin{aligned} L'_x &= y - \lambda \cdot 2x = 0 \\ L'_y &= x - \lambda \cdot 2y = 0 \\ x^2 + y^2 &= 10 \end{aligned}$$

$\left. \begin{aligned} &\text{FOC} = \\ &\text{first order conditions} \end{aligned} \right\}$
 $\left. \begin{aligned} &C = \text{constraint} \end{aligned} \right\}$

FOC + C = Lagrange conditions

Points $(x, y; \lambda)$ satisfying FOC + C: candidate pts

Then Necessary conditions for Lagrange problems

If $(x_1^*, x_2^*, \dots, x_n^*)$ is a solution to the Lagrange problem, and $(x_1^*, \dots, x_n^*)$ satisfies NDCQ, then there are Lagrange multipliers $\lambda_1, \dots, \lambda_m$ such that $(x_1^*, x_2^*, \dots, x_n^*; \lambda_1, \dots, \lambda_m)$ satisfies FOC + C.

Problems: i) We have to check NDCQ
ii) are necessary conditions

~~can~~
max/min $\Rightarrow$ candidate pt. (satisfying FOC + C)
or
pt. where NDCQ fails

Ex: min $f = xy$ when $g(x, y) = x^2 + y^2 = 10$

$$L = xy - \lambda(x^2 + y^2)$$

$$L'_x = y - \lambda \cdot 2x = 0$$

$$L'_y = x - \lambda \cdot 2y = 0$$

$$x^2 + y^2 = 10$$

$$y = \frac{2\lambda x}{1}$$

$$x - 2\lambda y = x - 2\lambda(2\lambda x) = 0$$

$$x - 4\lambda^2 x = 0$$

$$x \cdot (1 - 4\lambda^2) = 0$$

$$\underline{x=0} \text{ or } \lambda^2 = 1/4$$

$$\underline{\lambda = \pm 1/2}$$

c)

$$\lambda = -1/2$$

$$y = -x$$

$$2\lambda^2 = 10$$

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$

$$(x, y; \lambda) = (\sqrt{5}, -\sqrt{5}; -1/2)$$

$$(-\sqrt{5}, \sqrt{5}; -1/2)$$

$$\underline{f = -5}$$

a) $x=0$
 $y=0$
 $x^2 + y^2 = 0$
 $\neq 10$

no solutions

b) $\lambda = 1/2$
 $y = x$
 $2x^2 = 10$
 $x^2 = 5$
 $x = \pm\sqrt{5}$

$$(x, y; \lambda) = (\sqrt{5}, \sqrt{5}; 1/2)$$

$$(-\sqrt{5}, -\sqrt{5}; 1/2)$$

$$f = 5$$

Conclusion:

Candidate pts:
(FOC+C)

$(\sqrt{5}, \sqrt{5}; 1/2), (-\sqrt{5}, -\sqrt{5}; 1/2), (\sqrt{5}, -\sqrt{5}; -1/2), (-\sqrt{5}, \sqrt{5}; -1/2)$
 $f=5 \quad f=5 \quad f=-5 \quad f=-5$

best candidates for min

EVT: $x^2 + y^2 \leq 10$ compact $\Rightarrow$ There is a min

no pts where NDCQ fails ← Assume this

$f_{\min} = -5$ at $(\sqrt{5}, -\sqrt{5})$,
is minimum value $(-\sqrt{5}, \sqrt{5})$
with $\lambda = -1/2$

Ex: $\max/\min \quad f(x, y, z, w) = xw - yz \quad \text{w.h.m.} \quad \begin{cases} \overbrace{x^2 + y^2}^{g_1} = 16 \\ \underbrace{4z^2 + 9w^2}_{g_2} = 36 \end{cases}$

$$\begin{aligned} L &= f(x,y,z,w) - \lambda_1 \cdot g_1(x,y,z,w) - \lambda_2 \cdot g_2(x,y,z,w) \\ &= xw - yz - \lambda_1 (x^2 + y^2) - \lambda_2 (4z^2 + 9w^2) \end{aligned}$$

FOC $\left. \begin{aligned} L'_x &= w - 2\lambda_1 x = 0 \\ L'_y &= -z - 2\lambda_1 y = 0 \\ L'_z &= -y - 8\lambda_2 z = 0 \\ L'_w &= x - 18\lambda_2 w = 0 \end{aligned} \right\} \text{4, 2} \right\} x, w$

c $\left. \begin{aligned} x^2 + y^2 &= 16 \\ 4z^2 + 9w^2 &= 36 \end{aligned} \right\}$

x, w : $w - 2\lambda_1 x = 0$ $w = 2\lambda_1 x$
 $x - 18\lambda_2 w = 0$ $x - 18\lambda_2(2\lambda_1 x) = 0$
 $x - 36\lambda_1\lambda_2 x = 0$
 $x(1 - 36\lambda_1\lambda_2) = 0$

c) $\boxed{x=0, w=0}$ or d) $\boxed{\lambda_1\lambda_2 = 1/36, w=2\lambda_1 x}$

y, z : $-z - 2\lambda_1 y = 0$ $z = -2\lambda_1 y$
 $-y - 8\lambda_2 z = 0$ $-y - 8\lambda_2(-2\lambda_1 y) = 0$
 $-y + 16\lambda_1\lambda_2 y = 0$
 $y(-1 + 16\lambda_1\lambda_2) = 0$

c) $\boxed{y=0, z=0}$ or d) $\boxed{\lambda_1\lambda_2 = 1/16, z = -2\lambda_1 y}$

$x^2 + y^2 = 16$
 $4z^2 + 9w^2 = 36$

a) + c): $x = y = z = w = 0$
 not possible

a) + d): $x = w = 0, \lambda_1\lambda_2 = 1/16, z = -2\lambda_1 y$
 $y^2 = 16, 4z^2 = 36, -2\lambda_1 = \frac{z}{y} = \pm 3$
 $y = \pm 4, z = \pm 3, -2\lambda_1 = \pm 3/4$

$\lambda_1 = \mp 3/8$
 $\lambda_2 = \mp 1/16$

Candidates:

$f = -12$ $(0, 4, 3, 0; -3/8, -1/16)$
 $f = 12$ $(0, 4, -3, 0; 3/8, 1/16)$
 $f = 12$ $(0, -4, 3, 0; 3/8, 1/16)$
 $f = -12$ $(0, -4, -3, 0; -3/8, -1/16)$

b) + c): Similar with a) + d)

b) + d): no pts $\lambda_1\lambda_2 = 1/36, \lambda_1\lambda_2 = 1/16$

Plan:

① Example from Part 1 (continued)

② Kuhn-Tucker problems

① Continued: b) + c)

$$y = z = 0, \lambda_1 z = 1/36, w = 2\lambda_1 x$$

$$x^2 = 16 \quad 9w^2 = 36$$

$$x = \pm 4 \quad w = \pm 2$$

$$2\lambda_1 = \frac{w}{x} = \frac{\pm 2}{\pm 4} = \pm 1/2$$

$$\lambda_1 = \pm 1/4$$

$$\lambda_2 = \pm 1/9$$

Candidates:

$$f=8 \quad (4, 0, 0, 2; 1/4, 1/9)$$

$$f=-8 \quad (4, 0, 0, -2; -1/4, -1/9)$$

$$f=-8 \quad (-4, 0, 0, 2; -1/4, -1/9)$$

$$f=8 \quad (-4, 0, 0, -2; 1/4, 1/9)$$

$$f = xw - yz$$

<u>Best cand:</u>	max	$(0, 4, -3, 0), (0, -4, 3, 0)$	$\lambda_1 = 3/8, \lambda_2 = 1/6$	$f = 12$
	min	$(0, 4, 3, 0), (0, -4, -3, 0)$	$\lambda_1 = -3/8, \lambda_2 = -1/6$	$f = -12$

EVT:

$$x^2 + y^2 = 16$$

$$4z^2 + 9w^2 = 36$$

is compact

$$\frac{x^2}{16} + \frac{w^2}{9} = 1$$

(closed for Lagrange problem)

$$\text{bounded: } -4 \leq x \leq 4$$

$$-4 \leq y \leq 4$$

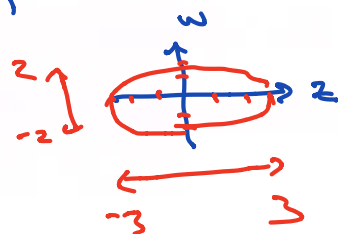
$$-3 \leq z \leq 3$$

$$-2 \leq w \leq 2$$

There is a max and min

Assume KKT satisfied

$$f_{\max} = \underline{\underline{12}} \quad f_{\min} = \underline{\underline{-12}}$$



② Kuhn - Tucker problems

Kuhn - Tucker problem:

Constrained
Optimization
problem

all constraints are
closed inequalities
($\leq$ or $\geq$)

$$\max/\min f(x_1, \dots, x_n) \quad \text{wh} \quad \begin{cases} g_1(x_1, \dots, x_n) \leq / \geq a_1 \\ \vdots \\ g_m(x_1, \dots, x_n) \leq / \geq a_m \end{cases}$$

KT-problem in standard form:

$$\max f(x_1, \dots, x_n) \quad \text{wh} \quad \begin{cases} g_1(x_1, \dots, x_n) \leq a_1 \\ \vdots \\ g_m(x_1, \dots, x_n) \leq a_m \end{cases}$$

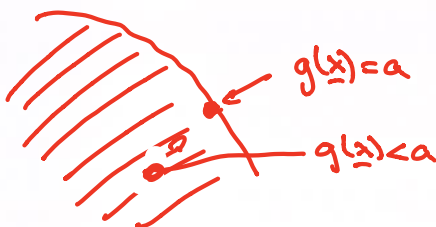
Trick: $\min f(x) = \max -f(x)$

$\cup$ f

$\cup$ $-f$

$$\begin{cases} x^2 + y^2 \geq 4 & 1 \cdot (-1) \\ -x^2 - y^2 \leq -4 \end{cases}$$

Method:



$$\max f(x_1, \dots, x_n) \quad \text{wh} \quad n \text{ variables}$$

$$\begin{cases} g_1(x_1, \dots, x_n) \leq a_1 \\ \vdots \\ g_m(x_1, \dots, x_n) \leq a_m \end{cases} \quad m \text{ inequalities}$$

KT problem in std form

$$L = f(x_1, \dots, x_n) - \lambda_1 g_1(x_1, \dots, x_n) - \lambda_2 g_2(x_1, \dots, x_n) - \dots - \lambda_m g_m(x_1, \dots, x_n)$$

$$\text{FOC: } \begin{cases} L'_{x_1} = 0 & f'_{x_1} - \lambda_1 g'_{1x_1} = 0 \\ L'_{x_2} = 0 \\ \vdots \\ L'_{x_n} = 0 \end{cases}$$

$$\text{c: } \begin{cases} g_1(x) \leq a_1 \\ \vdots \\ g_m(x) \leq a_m \end{cases}$$

CSC:
complementary
slackness
conditions

$$\begin{cases} \lambda_1 \geq 0 \\ \lambda_2 \geq 0 \\ \vdots \\ \lambda_m \geq 0 \end{cases}$$

and

Constraint i is binding if
 $g_i(x) = a_i$
and non-binding if
 $g_i(x) < a_i$

$$\begin{cases} \lambda_1 = 0 & \text{if } g_1(x) < a_1 \\ \lambda_2 = 0 & \text{if } g_2(x) < a_2 \\ \vdots \\ \lambda_m = 0 & \text{if } g_m(x) < a_m \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_1 (g_1(x) - a_1) = 0 \\ \lambda_2 (g_2(x) - a_2) = 0 \\ \vdots \\ \lambda_m (g_m(x) - a_m) = 0 \end{cases}$$

Kuhn-Tucker conditions:FOC + C + CSC

Pts $(x_1, \dots, x_n; \lambda_1, \dots, \lambda_m)$ that satisfy FOC + C + CSC
are candidate pts.

Ex: $\max f = xy$ when $x^2 + y^2 \leq 10$

KT-problem
std. form

$$L = xy - \lambda(x^2 + y^2)$$

$$\text{FOC} \begin{cases} L'_x = y - \lambda \cdot 2x = 0 \\ L'_y = x - \lambda \cdot 2y = 0 \end{cases}$$

$$C \quad x^2 + y^2 \leq 10$$

$$\text{CSC} \begin{cases} \lambda \geq 0 \\ \lambda(x^2 + y^2 - 10) = 0 \quad \checkmark \end{cases}$$

KT conditions
FOC + C + CSC

Best cand: $(\sqrt{5}, \sqrt{5}), (-\sqrt{5}, -\sqrt{5})$ $\lambda = 1/2$
 $f = 5$ $f = 5$

Binding

Non-binding

$$\begin{cases} y - 2\lambda x = 0 \\ x - 2\lambda y = 0 \\ x^2 + y^2 = 10 \end{cases}$$

$$\begin{cases} y - 2\lambda x = 0 \quad \checkmark \\ x - 2\lambda y = 0 \quad \checkmark \end{cases}$$

$$x^2 + y^2 < 10 \quad \text{ok.}$$

$$\lambda \geq 0$$

$$\lambda = 0 \quad \checkmark$$

$$(x, y; \lambda) =$$

$$(\sqrt{5}, \sqrt{5}; 1/2) \quad f = 5$$

$$(-\sqrt{5}, -\sqrt{5}; 1/2) \quad f = 5$$

$$(\sqrt{5}, -\sqrt{5}; -1/2)$$

$$(-\sqrt{5}, \sqrt{5}; -1/2)$$

$$\begin{aligned} y &= 0 \\ x &= 0 \end{aligned}$$

$$(x, y; \lambda) = (0, 0; 0) \quad f = 0$$

EVT:

$$x^2 + y^2 \leq 10$$

closed ($\leq$)bounded $\checkmark$ compact

$$\begin{cases} -\sqrt{10} \leq x \leq \sqrt{10} \\ -\sqrt{10} \leq y \leq \sqrt{10} \end{cases}$$

 $\Rightarrow$ there is a max $\Downarrow$ Assume
NOCB holds

$$f_{\max} = 5 \quad \text{at } (x, y) = (\sqrt{5}, \sqrt{5}), (-\sqrt{5}, -\sqrt{5}) \quad \text{with } \lambda = 1/2$$

Thm (Necessary conditions for KT problems)

If $(x_1^*, \dots, x_n^*)$ solves the KT problem, and if NDCQ is satisfied, then there exists $\lambda_1, \dots, \lambda_m$ such that $(x_1^*, \dots, x_n^*; \lambda_1, \dots, \lambda_m)$ satisfy FOC + C + CSC.

$$(x_1^*, \dots, x_n^*) \max \Rightarrow \left\{ \begin{array}{l} (x_1^*, \dots, x_n^*; \lambda_1, \dots, \lambda_m) \text{ satisfy FOC + C + CSC} \\ \text{or} \\ (x_1^*, \dots, x_n^*) \text{ fails NDCQ} \end{array} \right.$$

Reminder: Plenary Session 2
Monday 17-20

tell me if there are
pb's for Lectur 4-6
you want me to go
through