Review: Retake exam in GRA6035
March 2021

$D_1 = 1$, $D_2 = 0$, $D_3 = 0$, $D_4 = 0$ $\qquad$ $\text{rk}\, A = 3$

1 c) $f(\underline{x}) = \underline{x}^T A \underline{x}$ $\qquad A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 10 \end{pmatrix}$

$= (x\ y\ z\ w) \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ not symmetric

$= x^2 + 4xy + 4y^2 + 2yz + 2xw + 5z^2 + 12zw + 10w^2$

$= \underline{x}^T \cdot B \underline{x}$, with $B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 0 \\ 0 & 1 & 5 & 6 \\ 1 & 0 & 6 & 10 \end{pmatrix}$

$D_1 > 0$

$D_1 = 1$
$D_2 = 4 - 4 = 0$
$D_3 = 5 \cdot 0 - 1 \cdot 1 = -1$ $\qquad D_3 < 0$ indefinite

d) max/min $\underline{x}^T A \underline{x}$ when $\underline{x}^T \cdot \underline{x} = 1$

Lagrange problem: max/min $f(x,y,z,w) = x^2 + 4xy + \dots$
when $x^2 + y^2 + z^2 + w^2 = 1$

$\underline{x}^T \underline{x} = x^T \cdot I \cdot \underline{x}$
$= (x\ y\ z\ w) \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$
$= x^2 + y^2 + z^2 + w^2$

$\mathcal{L} = (x^2 + 4xy + \dots + 10w^2) - \lambda(x^2 + y^2 + z^2 + w^2)$

$\mathcal{L}'_x = 2x + 4y + 2w - \lambda \cdot 2x = 0$
$\mathcal{L}'_y = 4x + 8y + 2z - \lambda \cdot 2y = 0$
$\mathcal{L}'_z = 2y + 10z + 12w - \lambda \cdot 2z = 0$
$\mathcal{L}'_w = 2x + 12z + 20w - \lambda \cdot 2w = 0$

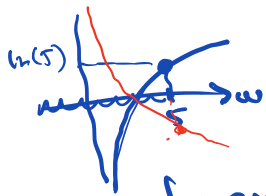
$x^2 + y^2 + z^2 + w^2 = 1$

$$\begin{pmatrix} 2-2\lambda & 4 & 0 & 2 \\ 4 & 8-2\lambda & 2 & 0 \\ 0 & 2 & 10-2\lambda & 12 \\ 2 & 0 & 12 & 20-2\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2\. a) max $f(x,y,z) = \ln(5 - x^2 + xy - y^2 + yz - z^2 - xz)$

$= \ln(w)$, where $w = 5 - x^2 + xy \cdots - xz$

max/min w	$h(w) = \ln(w)$
$w = 5 + \underline{x}^T \cdot \begin{pmatrix} -1 & 1/2 & -1/2 \\ 1/2 & -1 & 1/2 \\ -1/2 & 1/2 & -1 \end{pmatrix} \underline{x}$	$h'(w) = \frac{1}{w} > 0$
	$\ln$ is increasing



$D_1 = -1 < 0$

$D_2 = 1 - 1/4 = 3/4 > 0$

$D_3 = -1 \cdot (3/4)$
$- 1/2(-1/2 - 1/4)$
$- 1/2(1/4 - 1/2)$
$= -\frac{3}{4} + \frac{3}{8} + \frac{1}{8} = -\frac{1}{4} < 0$

neg. def.

w has max value $w = 5$ at $\underline{x} = (0,0,0)$

$\boxed{w \leq 5}$

$f_{max} = \ln(5)$

b) $D: xw + yz \leq -2$ compact?

✓ compact = closed + bounded

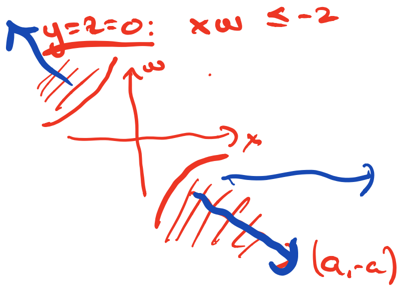
✓ D is closed since given by $\leq$

Bounded:

$x = 100$, $w = -100$, $y = 0$, $z = 0$

$\underbrace{xw - yz}_{} = -10{,}000$

$y = z = 0$: $xw \leq -2$



$x = a,\ w = -a,\ y = z = 0$: $xw + yz = a \cdot (-a) = -a^2 \leq -2$

ok if $a^2 \geq 2$

Hence: $(a,0,0,-a) \in D$ for all $a > \sqrt{2}$

$f_t = 4t \quad \ldots) = 4t + 4 \ldots 8$

3a) $y_{t+2} - y_{t+1} - 2y_t = 4t$

$f_{t+1} = 4(t+1)$

$f_{t+2} = 4(t+2) = 4t + \ldots$

$$y_t = y_t^h + y_t^p = \underbrace{c_1 \cdot 2^t + c_2 \cdot (-1)^t}\ \underbrace{-2t-1}$$

y_t^h: $y_{t+2} - y_{t+1} - 2y_t = 0$

$$r^2 - r - 2 = 0$$

$$r = \frac{1 \pm \sqrt{1^2 - 4\cdot(-2)}}{2}$$

$$= \frac{1 \pm 3}{2} = 2, -1$$

$$y_t^h = c_1 \cdot 2^t + c_2 \cdot (-1)^t$$

y_t^p: $y_t = At + B$

$$y_{t+1} = A(t+1) + B = At + A + B$$

$$y_{t+2} = A(t+2) + B = At + 2A + B$$

$$(\cancel{At} + 2A + \cancel{B})$$

$$-(\cancel{At} + A + \cancel{B})$$

$$-2(At + B) = 4t$$

$$-2A \cdot t + A - 2B = 4t + 0$$

$$-2A = 4 \quad A = -2$$

$$A - 2B = 0$$

$$-2 - 2B = 0 \quad B = -1$$

$$y_t^p = -2t - 1$$

$$y_t = c_1 \cdot 2^t + c_2 \cdot (-1)^t - 2t - 1$$

$$y_0 = c_1 + \cancel{c_2} - 1 = 1$$

$$y_1 = 2c_1 - \cancel{c_2} - 3 = 1$$

$$3c_1 = 2 + 4$$

$$c_1 = \frac{6}{3} = 2$$

$$c_2 = 0$$

$$y_t = 2 \cdot 2^t - 2t - 1$$

$$y_{17} = 2 \cdot 2^{17} - 2 \cdot 17 - 1$$

$$y'' - y' - 2y = 0 \qquad y = c_1 \cdot e^{2t} + c_2 \cdot e^{-t}$$

$$y_{t+2} - y_{t+1} - 2y_t = 0 \qquad y = c_1 \cdot 2^t + c_2 \cdot (-1)^t$$

3c. $y = \underbrace{3e^{-2t} - 5e^{t}}_{y_h} + \underbrace{12e^{-3t}}_{y_p}$.

Solution: $y_h + y_p$

$y'' + ay' + by = 0$

$r^2 + ar + b = 0$

two roots r_1, r_2

$y_h = C_1 \cdot e^{r_1 t} + C_2 e^{r_2 t}$

$y'' + y' - 2y = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r_1 = -2$

$r_2 = 1$

Differential equation: $y'' + y' - 2y = 48e^{-3t}$

$y_p = 12e^{-3t}$ $\quad y'' + y' - 2y = 9 \cdot 12e^{-3t} + (-3) 12e^{-3t} - 2 \cdot 12e^{-3t}$

$= 108e^{-3t} - 36e^{-3t} - 24e^{-3t}$

$= 48e^{-3t}$

Linear: $y' + a(t)y = b(t)$

Separable: $y' = f(t) \cdot g(y)$

Exact: $p(t,y) + q(t,y)y' = 0$

s.t. $h'_t = p$ and

$h'_y = q$

$t^2y' + 2ty = 1 \quad | : t^2$

$y' + \frac{2t}{t^2} y = \frac{1}{t^2}$

$\frac{t^2 y'}{t^2} = \frac{1 - 2ty}{t^2}$

$y' = \frac{1 - 2ty}{t^2}$

$h = t^2y - t \;\checkmark$ $\qquad \underbrace{t^2}_{= q} y' + \underbrace{2ty - 1}_{= p} = 0$