	Plan	
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① Key problems

10.4 c)
$$\frac{y(1-2\ln t)}{t^3} + \frac{\ln t}{t^2} y' = 0$$

$$\boxed{p + q \cdot y' = 0}$$
exact?
$$p = h'_t, \quad q = h'_y \; ?$$
If so, then it is exact and $h = C$ is the general solution.

$$h'_t = \frac{y(1-2\ln t)}{t^3}$$

$$h'_y = \frac{\ln t}{t^2} \leftarrow \text{start with this one}$$

$$h = \frac{\ln t}{t^2} \cdot y + \alpha(t)$$

$\Downarrow$

$$h'_t = \left(\frac{\ln t}{t^2} \cdot y + \alpha(t)\right)'_t$$
$$= y \cdot \frac{\frac{1}{t} \cdot t^2 - \ln t \cdot (2t)}{t^4} + \alpha'(t)$$
$$= \frac{y \cdot (t - 2t\ln t)}{t^4} + \alpha'(t)$$
$$= y\,\frac{(1-2\ln t)}{t^3} + \alpha'(t) = \frac{y \cdot (1-2\ln t)}{t^3}$$

$\alpha'(t) = 0$ ok choose $\alpha = 0$

Exact: $h = \frac{\ln t}{t^2} \cdot y = C$
$$y = \frac{Ct^2}{\ln(t)}$$

11.1 c) $y' = 5y(1 - y/10)$

$y' = F(y)$

separable
lin. diff. eqn.
autonomous

Eq. states: $F(y) = 0$

$5y(1 - y/10) = 0$

$5y = 0$ or $1 - y/10 = 0$

$\underline{y=0}$ or $\underline{y=10}$

$\underline{\underline{y_e = 0}}$, $\underline{\underline{y_e = 10}}$ are the eq. states.

Stability thm:

$F'(y) = 5 - y$

$F'(10) = -5 < 0 \Rightarrow$ stable

$F'(0) = 5 > 0 \Rightarrow$ unstable

Stability:

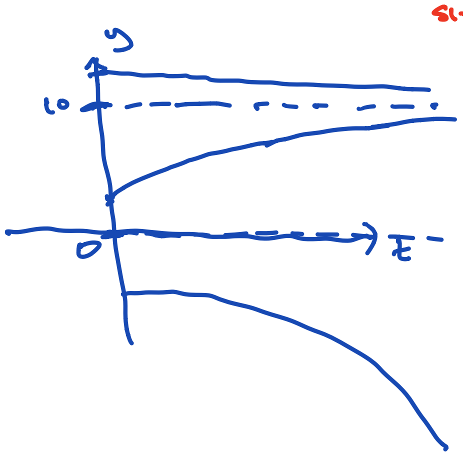
$y_e = 10$ is $\underline{\underline{\text{stable}}}$

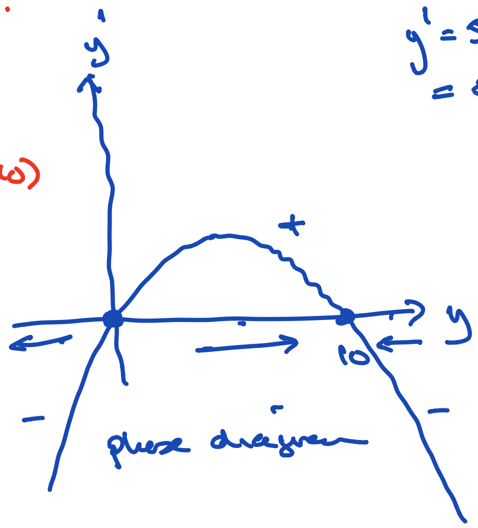
$y_e = 0$ is $\underline{\underline{\text{unstable}}}$

$y_e = 10$ not globally as. stable

since $y_0 < 0$ gives $y(t) \to -\infty$ (away from $y=10$)

$y' = 5y(1 - y/10)$
$= 5y - \frac{5y^2}{10}$





12.3 $\quad \underline{y_{t+1}} = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} y_t$

$y_t = \begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{pmatrix}$

$A = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix}$:

Eigenvalues: $\begin{vmatrix} -5-\lambda & 0 & 1 \\ 0 & -3-\lambda & 0 \\ 1 & 0 & -5-\lambda \end{vmatrix} = 0$

$(-3-\lambda) \cdot \begin{vmatrix} -5-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = 0$

$(-3-\lambda)(\lambda^2 + 10\lambda + 24) = 0$

$\lambda_1 = \underline{-3}$, $\lambda_2 = \underline{-4}$, $\lambda_3 = \underline{-6}$

Since A is diagonalizable, the general solution is

$$\boxed{y_t = C_1 \cdot \underline{v_1} \cdot \lambda_1^t + C_2 \cdot \underline{v_2} \cdot \lambda_2^t + C_3 \cdot \underline{v_3}\, \lambda_3^t}$$

$z_{t+1} = \lambda_i z_t$:

$z_1 = \lambda \cdot z_0$

$z_2 = \lambda^2 \cdot z_0$

$\vdots$

$z_t = \lambda^t \cdot z_0$

$\lambda_1 = -3$: $\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $x = 2z$, $-3z = 0$, y free; $x = 0$, $z = 0$

$\underline{v} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -4$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ \cancel{1} & \cancel{0} & \cancel{-1} \end{pmatrix}$ $x = z$, $y = 0$, z free $\quad \underline{v} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3 = -6$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{pmatrix}$ $x + z = 0$, $3y = 0$, z free; $x = -z$, $y = 0$ $\quad \underline{v} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Solution:

$$y_t = c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (-3)^t + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot (-4)^t + c_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot (-6)^t$$

$$= \begin{pmatrix} c_2 \cdot (-4)^t - c_3 (-6)^t \\ c_1 \cdot (-3)^t \\ c_2 \cdot (-4)^t + c_3 (-6)^t \end{pmatrix}$$

2 Problems from [E]

9.14. $y' = \underbrace{r}_{f(t)} \underbrace{y(1 - y/K)}_{g(y)}$ $\quad r, K$ const. with $K > 0$

separable

$$\frac{1}{y(1 - y/K)} y' = r \quad \begin{matrix} \cdot K \\ \cdot K \end{matrix}$$

$$\frac{K}{y(K-y)} y' = r \quad \Big| \int \ldots dt$$

$$\int \frac{K}{y(K-y)} dy = \int r \, dt = rt + C$$

Partial fraction:

$$\frac{K}{y \cdot (K-y)} = \frac{A}{y} + \frac{B}{K-y} \quad \text{1.cd}$$

$$K = A(K-y) + By$$

$$K = \underbrace{(AK)}_{=K} + \underbrace{(B-A)y}_{=0}$$

$A = 1 \quad B = A$

$$\boxed{\frac{K}{y(K-y)} = \frac{1}{y} + \frac{1}{K-y}}$$

$$\int \frac{1}{y} + \frac{1}{K-y}\,dy = rt + C$$

$$\ln|y| - \ln|K-y| = rt + C$$

$$\ln\left(\frac{|y|}{|K-y|}\right) = rt + C \quad | e^{\cdot}$$

$$\frac{|y|}{|K-y|} = e^{rt+C} = e^{rt}\cdot e^{C}$$

$$\frac{y}{K-y} = \boxed{\pm e^{C}}\cdot e^{rt} = A\cdot e^{rt}$$

$$\frac{y}{K-y} = Ae^{rt} \quad |\cdot(K-y)$$

$$y = Ae^{rt}(K-y)$$

$$y = KAe^{rt} - y\cdot Ae^{rt}$$

$$y + y\cdot Ae^{rt} = KAe^{rt}$$

$$y\cdot\frac{(1+Ae^{rt})}{1+Ae^{rt}} = \frac{KAe^{rt}}{1+Ae^{rt}}$$

$$y = K\cdot\frac{Ae^{rt}}{1+Ae^{rt}}$$

9.1e

$$t^2y' + \ln(t)y = \ln(t) \quad |:t^2$$

linear first order, solve by superposition

$$y' + \underbrace{\frac{\ln t}{t^2}}_{a(t)}\cdot y = \underbrace{\frac{\ln t}{t^2}}_{b(t)}$$

$$y = y_h + y_p =$$

y_h:

$$y' + \frac{\ln t}{t^2}y = 0$$

$$y' = -\frac{\ln t}{t^2}y$$

$$\frac{1}{y}y' = -\frac{\ln t}{t^2}$$

$$\int\frac{1}{y}dy = \int -\frac{\ln t}{t^2}dt$$

Int. by parts: $\int u'v\,dt = uv - \int uv'\,dt$

$$\boxed{\begin{matrix} u = -t^{-1} & v = \ln t \\ u' = t^{-2} & v' = 1/t \end{matrix}}$$

$$\int -\frac{1}{t^2}\cdot\ln t\,dt$$

$$= -\frac{1}{t}\ln(t) - \int t^{-1}\cdot t^{-1}dt$$

$$= -\frac{1}{t}\ln t - \int t^{-2}dt$$

$$= -\frac{1}{t}\ln t + \frac{1}{t} + C$$

$$\ln|y| = -\frac{1}{t}\ln t + \frac{1}{t} + C$$

$$|y| = e^{-\frac{1}{t}\ln t + \frac{1}{t} + C} = e^{-\frac{1}{t}\ln t + \frac{1}{t}}\cdot e^{C}$$

$$y = (\pm e^{C})\, e^{\frac{1}{t}(1-\ln t)}$$

$$y_h = Ke^{\frac{1}{t}(1-\ln t)}$$

$\underline{y_p}$: $y' + \frac{\ln t}{t^2} y = \frac{\ln t}{t^2}$

Try constant solutions:

$y_p = 1$

$$y = y_h + y_p = \underline{\underline{K \cdot e^{\frac{1}{t}(1-\ln t)} + 1}}$$

③ Problems [WB]

$\dot{y} = y'$

$y(\sqrt{2}) = 1$

10.8 c) $y y' = t$ $\boxed{(t_0, y_0) = (\sqrt{2}, 1)}$

$y' = t \cdot \frac{1}{y}$

$y y' = t \quad | \int - dt$

$\int y y' dt = \int t\,dt$

$\int y\,dy = \int t\,dt$

$\frac{1}{2} y^2 = \frac{1}{2} t^2 + C \quad | \cdot 2$

$y^2 = t^2 + 2C$

$y = \pm \sqrt{t^2 + 2C}$

$= \pm \sqrt{t^2 - 1}$

$1^2 = (\sqrt{2})^2 + 2C$

$1 = 2 + 2C$

$-1 = 2C \quad C = -\frac{1}{2}$

$y = \underline{\underline{\sqrt{t^2 - 1}}}$ since $y = -\sqrt{t^2-1}$ gives $y(\sqrt{2}) = -\sqrt{2-1} = -1$

d) $e^{2t} \cdot y' - y^2 - 2y = 1 \quad , \quad y(0) = 0$

$e^{2t} \cdot y' = y^2 + 2y + 1 = (y+1)^2$

$y' = \underbrace{e^{-2t}}_{f(t)} \underbrace{(y+1)^2}_{g(y)}$

separable

$\frac{1}{(y+1)^2} y' = e^{-2t} \quad | \int - dt$

$\int \frac{1}{(y+1)^2} dy = \int e^{-2t} dt$

$-\frac{1}{y+1} = \frac{1}{-2} \cdot e^{-2t} + C \quad | \cdot (-1)$

$\boxed{\begin{array}{l} u = y+1 \\ du = dy \end{array}}$

$\int \frac{1}{(y+1)^2} dy = \int \frac{1}{u^2} du$

$= -u^{-1} + C = -\frac{1}{y+1} + C$

$$\frac{1}{y+1} = \frac{1}{2}e^{-2t} + C \quad | \cdot (y+1) \cdot 2$$

$$\frac{2}{e^{-2t}+2C} = e^{-2t}(y+1) + 2C(y+1) = \frac{(\cancel{e^{-2t}+2C})(y+1)}{\cancel{e^{-2t}+2C}}$$

$$y+1 = \frac{2}{e^{-2t}+2C} \cdot \frac{e^{2t}}{e^{2t}} = \frac{2e^{2t}}{1+2Ce^{2t}}$$

$$y = \frac{2e^{2t}}{1+2Ce^{2t}} - 1 = \frac{2e^{2t} - (1+2Ce^{2t})}{1+2Ce^{2t}}$$

$$y = \frac{(2-2C)e^{2t}-1}{2Ce^{2t}+1}$$

$y(0)=0$:

$$0 = \frac{(2-2C)\cdot 1 - 1}{2C\cdot 1 + 1} = \frac{1-2C}{1+2C} \Rightarrow C = 1/2$$

$$\Rightarrow \quad y = \frac{e^{2t}-1}{e^{2t}+1}$$

4.8. a) $t^2x'' + 5tx' + 3x = 0$ linear second order, not const coeff.

Try $x = t^r$:

$$\begin{cases} x = t^r \\ x' = r\cdot t^{r-1} \\ x'' = r\cdot(r-1)t^{r-2} \end{cases}$$

$$t^2 \cdot r(r-1)t^{r-2} + 5t\cdot r\cdot t^{r-1} + 3\cdot t^r = 0$$

$$r\cdot(r-1)\cdot t^r + 5rt^r + 3t^r = 0$$

$$t^r\cdot[r(r-1) + 5r + 3] = 0$$

$$t^r\cdot(r^2+4r+3) = 0$$

ok if $r^2+4r+3=0$

$r_1=-3,\ r_2=-1$

$$\Rightarrow x = C_1\cdot t^{-3} + C_2\cdot t^{-1} = C_1\cdot\frac{1}{t^3} + C_2\cdot\frac{1}{t}$$

12.7 $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$

second order linear difference equation (constant coeff.)

$x_t = x_t^h + x_t^p$

$= (C_1 + C_2 t) \cdot (-1)^t + 2^t$

x_t^h: $x_{t+2} + 2x_{t+1} + x_t = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r_1 = r_2 = -1$

$x_t^h = C_1 \cdot (-1)^t + C_2 \cdot t \cdot (-1)^t$

$= (C_1 + C_2 t) \cdot (-1)^t$

x_t^p: $x_{t+2} + 2x_{t+1} + x_t = 9 \cdot 2^t$

$f_t = 9 \cdot 2^t$

$f_{t+1} = 9 \cdot 2^{t+1} = 9 \cdot 2^t \cdot 2$

$= 18 \cdot 2^t$

$f_{t+2} = 36 \cdot 2^t$

$\Downarrow$

$4A \cdot 2^t + 2 \cdot 2A \cdot 2^t + A \cdot 2^t = 9 \cdot 2^t$ $\Leftarrow$ $\begin{cases} x_t = A \cdot 2^t \\ x_{t+1} = 2A \cdot 2^t \\ x_{t+2} = 4A \cdot 2^t \end{cases}$

$9A \cdot 2^t = 9 \cdot 2^t$

$A = 1$

$\Downarrow$

$x_t^p = 1 \cdot 2^t = 2^t$