## Alternative Solution: Final exam 12/2021, Q3 (c) – (d)

$f(\underline{x}) = x + y + z + w = B\underline{x}$ $\quad B = (1\ 1\ 1\ 1)$ $\quad \underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$g(\underline{x}) = 3x^2 + 2xy + 8xz - 2xw + y^2 + 4yz + 2yw + 7z^2 + 4w^2 = \underline{x}^T A \underline{x}$

$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$

(a) Quadr. form $g$ has symmetric matrix $A$

$D_1 = \underline{3}$ $\qquad D_3 = 3(7-4) - 1(7-8) + 4(2-4) = \underline{2}$

$D_2 = 3 - 1 = \underline{2}$ $\qquad D_4 = |A| = \dots = \underline{2}$ ← see published solution.

Conclusion: $A$ pos. defn. $\Rightarrow$ $g$ pos. defn.

(b) Kuhn–Tucker conditions: max $\underset{\substack{\| \\ B\underline{x}}}{f(\underline{x})}$ when $\underset{\substack{\| \\ \underline{x}^T A \underline{x}}}{g(\underline{x})} \leq 18$

KT problem in std. form

$\mathcal{L} = f(\underline{x}) - \lambda \cdot (g(\underline{x}) - 18)$

$= x + y + z + w - \lambda(3x^2 + 2xy + \dots + 4w^2 - 18)$

$= B\underline{x} - \lambda \cdot (\underline{x}^T A \underline{x} - 18)$

$$\boxed{\begin{array}{ll} \text{FOC:} & \mathcal{L}'(\underline{x}) = B^T - \lambda \cdot (2A\underline{x}) = 0 \\ \text{C:} & \underline{x}^T A \underline{x} \leq 18 \\ \text{CSC:} & \lambda \geq 0,\ \lambda(\underline{x}^T A \underline{x} - 18) = 0 \end{array}}$$

KT conditions in matrix form

See [E] Chap 5.6:

$f(\underline{x}) = B\underline{x} \Rightarrow f'(\underline{x}) = B^T$

$g(\underline{x}) = \underline{x}^T A \underline{x} \Rightarrow g'(\underline{x}) = 2A\underline{x}$

Alternative: KT conditions without

FOC:
$$\begin{cases} \mathcal{L}'_x = 1 - \lambda \cdot (6x + 2y + 8z - 2w) = 0 \\ \mathcal{L}'_y = 1 - \lambda\ (2x + 2y + 4z + 2w) = 0 \\ \mathcal{L}'_z = 1 - \lambda\ (8x + 4y + 14z) = 0 \\ \mathcal{L}'_w = 1 - \lambda\ (-2x + 2y + 8w) = 0 \end{cases}$$

Compare with matrix form:

$B^T = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\qquad 2A\underline{x} = \begin{pmatrix} 6x + 2y + 8z - 2w \\ \vdots \\ -2x + 2y + 8w \end{pmatrix}$

C: $3x^2 + 2xy + \dots + 4w^2 \leq 18$

CSC: $\lambda \geq 0,\ \lambda(3x^2 + 2xy + \dots + 4w^2 - 18) = 0$

c) NDCQ: Non-binding case: $g(\underline{x}) < 18$ no condition to check

Binding case: $g(\underline{x}) = 18$:

$J = (g'_x \ g'_y \ g'_z \ g'_w)$ $\boxed{\text{NDCQ: } rk\, J = 1}$

$g'_x = 6x+2y+8z-2w$, $g'_w = -2x+2y+8w$

NDCQ fails: $rk\, J < 1$, $rk\, J = 0$, $J = 0$

NDCQ fails:

$$
\left.\begin{aligned}
g'_x &= 6x+2y+8z-2w = 0 \\
g'_y &= \ldots = 0 \\
g'_z &= \ldots = 0 \\
g'_w &= -2x+2y+8w = 0
\end{aligned}\right\}
$$

Solve this: Lin system

$2A\cdot\underline{x} = \underline{0}$ | : 2

know that $|A| = 2 \neq 0$ from (a)

$A\underline{x} = \underline{0}$ | $A^{-1}$

$\underline{x} = \underline{0}$

NDCQ fails: $\underline{x} = \underline{0}$

$(x,y,z,w) = (0,0,0,0)$ ← This does not satisfy $g(\underline{x}) = 18$

Conclusion: NDCQ satisfied at all adm pts (in both non-binding and binding case).

(d) First find candidate pts:

With matrix form:

FOC: $B^T - \lambda\cdot(2A\underline{x}) = \underline{0}$

$-\lambda\cdot 2A\underline{x} = -B^T$ | : $(-\lambda)$

$2A\underline{x} = \frac{1}{\lambda}\cdot B^T$

$$
\underbrace{\begin{aligned}
6x+2y+8z-2w &= \tfrac{1}{\lambda} \\
2x+2y+4z+2w &= \tfrac{1}{\lambda} \\
8x+4y+14z &= \tfrac{1}{\lambda} \\
-2x+2y+4w &= \tfrac{1}{\lambda}
\end{aligned}}_{2A\cdot\underline{x}} \qquad \underbrace{\phantom{x}}_{\frac{1}{\lambda}\cdot B^T}
$$

Alternative:

$1 - \lambda(6x+2y+8z-2w) = 0$

$1 - \lambda(2x+2y+4z+2w) = 0$

$1 - \lambda(8x+4y+14z) = 0$

$1 - \lambda(-2x+2y+4w) = 0$

Same equations

- Solve using Gauss (linear system)
- Set $1/\lambda = a$ to simplify computations (less to write)

After Gauss: $(x,y,z,w) = (\frac{1}{2}a, a, -\frac{1}{2}a, 0)$

C+CSC: $\lambda = 0$ impossible

$\lambda > 0$, $g(\underline{x}) = 18 \Rightarrow g(\frac{1}{2}a, a, -\frac{1}{2}a, 0) = 18$

$\boxed{3\cdot(a/2)^2 + 2(a/2)a + \ldots = 18 \quad \frac{1}{2}a^2 = 18}$

$\frac{1}{2}a^2 = 18$

$a^2 = 36$ $a = \pm 6$ $\frac{1}{\lambda} = \pm 6 = 6 \Rightarrow \lambda = 1/6$ ($\lambda > 0$)

Candidate pt: $(x, y, z, w\, ;\, \lambda) = \underline{(3, 6, -3, 0\, ;\, 1/6)}$ with $a = 6$, $\lambda = 1/6$

Then check that candidate pt is max:

SOC: $h(x,y,z,w) = h(x,y,z,w; 1/6) = Bx - \frac{1}{6}(\underline{x}^T A \underline{x} - 18)$

$$= \underbrace{x + y + z + w}_{\text{Hessian} = 0} - \frac{1}{6}\Big(\underbrace{3x^2 + \cdots + 4w^2}_{\substack{\text{Hessian} \\ = 2A}} \underbrace{- 18}_{\text{Hessian} = 0}\Big)$$

$H(h) = -\frac{1}{6} \cdot 2A = -\frac{1}{3} A$

$A$ pos. defn (from a)

$\Rightarrow -\frac{1}{3}A$ neg. defn. $\Rightarrow h$ concave $\overset{SOC}{\Longrightarrow}$ $(3, 6, -3, 0)$ is max pt