

Plan

- 1 First order differential equations
- 2 Separable differential equations
- 3 Exact differential equations

Mon Nov 01: Plenary Session 3

Problems from Lecture 7-9

Review:Quadratic functions

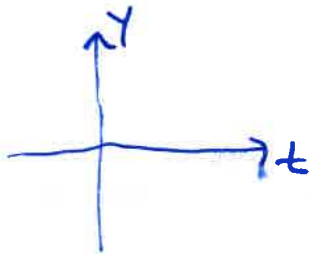
$$f(x) = x^T A x + Bx + C \Rightarrow f'(x) = 2Ax + B^T$$

A non symm. matrix B 1xn matrix C const.

$$H(x) = 2A$$

① First order differential equationsVariable:

$y = y(t)$
function in
one variable



$t = \text{time}$

$y(t)$: a quantity
 $y'(t)$: rate of change

Differential equation of order one

An equation involving $y'(t)$ but
not higher derivatives

Ex:

$$y' = 2y \Leftrightarrow y'(t) = 2 \cdot y(t)$$

$$y' + e^t y = 2\sqrt{t}$$

$$0 \Rightarrow y' = 2\sqrt{t} - e^t y$$

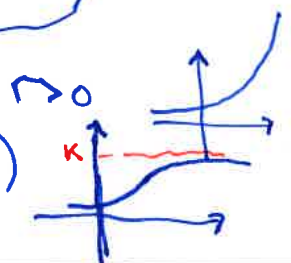
In general: $y' = F(t, y)$

Ex:

$$y' = r \cdot y$$

$$y' = r \cdot y \cdot (1 - y/K)$$

$$r > 0, K > 0$$



Ex: $y' = 2t$, $y(0) = 1$
 $\Downarrow$ initial condition

$$y = \int 2t dt = t^2 + C$$

General solution: $y(t) = t^2 + C$

$y(0) = 1$: $y(0) = 0^2 + C = 1$
 $C = 1$

$\Downarrow$

Particular solution: $y(t) = \underline{\underline{t^2 + 1}}$

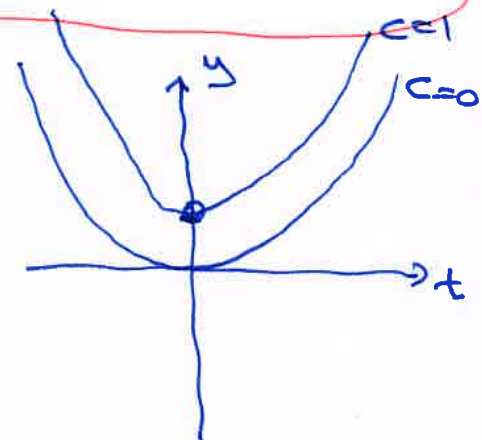
A first order differential equation has general solution that involves one undetermined coefficient. $\leftarrow$ general solution

A first order differential equation with one initial condition give a specific value of the undetermined coefficient.

$\nwarrow$ particular solution

Ex: General solution: $y(t) = t^2 + C$

Each value of $C \leftrightarrow$ one solution curve



② Seperable differential equations

Defn: A first order differential equation is seperable if it can be written

$$y' = f(t) \cdot g(y)$$

$y' = F(t, y)$
general
first order
diff. eqn

Ex: ① $y' = 2y = \underbrace{2}_{f(t)} \cdot \underbrace{y}_{g(y)} \leftarrow \text{seperable}$

② $y' = y + t \leftarrow \text{not separable}$

③ $y' = ry \quad (r \text{ const.})$

"exponential growth"
is seperable

④ $y' = ry(1 - y/K)$
(r, K const.)

"logistic growth model"
is seperable

Solution method:

$$\begin{aligned} y' &= \underbrace{2}_{f(t)} \underbrace{y}_{g(y)} \quad | : y \\ \frac{1}{y} \cdot y' &= 2 \quad | \int \dots dt \\ \int \frac{1}{y} y' dt &= \int 2 dt \\ \int \frac{1}{y} dy &= \int 2 dt \\ \ln |y| + C_1 &= 2t + C_2 \\ \ln |y| &= 2t + (C_2 - C_1) \\ \ln |y| &= 2t + K \end{aligned}$$

implicit
solution

$$\begin{aligned} y' &= f(t) \cdot g(y) \quad | : g(y) \\ \frac{1}{g(y)} \cdot y' &= f(t) \quad | \int \dots dt \end{aligned}$$

$$\int \frac{1}{g(y)} y' dt = \int f(t) dt$$

$$\boxed{\int \frac{1}{g(y)} dy = \int f(t) dt}$$

implicit solution

$y' dt = dy$: Substitution $y = y(t)$
 $dy = y' \cdot dt$

$$e^{\ln|y|} = e^{2t+K}$$

$$|y| = e^{2t} \cdot e^K$$

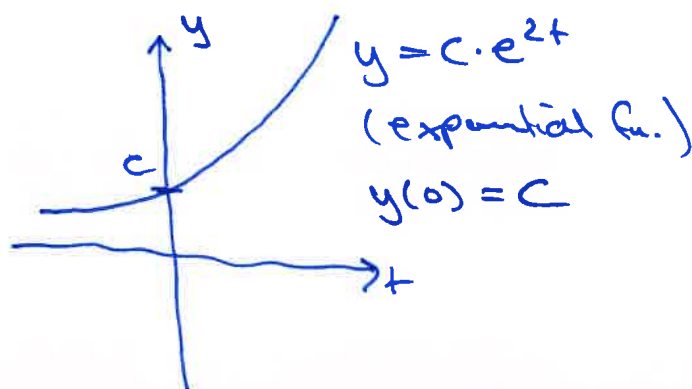
$$y = \pm e^{2t} \cdot e^K$$

$$\underline{y = C \cdot e^{2t}}$$

(explicit form)

general solution

$$(C = \pm e^K)$$



Ex:

$$e^{2y} \cdot y' = t e^t$$

$$y' = \underbrace{t e^t}_{f(t)} \cdot \underbrace{e^{-2y}}_{g(y)}$$

$$e^{2y} \cdot y' = t e^t$$

$$\int e^{2y} y' dt = \int t e^t dt$$

$$\int e^{2y} dy = \int t e^t dt \leftarrow \text{integration by parts}$$

$$\frac{1}{2} e^{2y} = t e^t - \int e^t dt$$

$$\frac{1}{2} e^{2y} = t e^t - e^t + C \quad | \cdot 2$$

$$e^{2y} = 2 t e^t - 2 e^t + 2C$$

$$\frac{2y}{2} = \ln(2 t e^t - 2 e^t + 2C) = \ln(2 \cdot (t e^t - e^t + C))$$

$$y = \frac{\ln[2 \cdot (t e^t - e^t + C)]}{2} = \frac{\ln 2 + \ln(t e^t - e^t + C)}{2}$$

$$y' = f(t) \cdot g(y)$$

separable

$u = t e^t \quad v = t$ $u' = e^t \quad v' = 1$
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$$\int u' v dt = uv - \int u v' dt$$

② Exact differential equations

Defn:

A first order differential equation is exact if it can be written

$$p(t,y) + q(t,y) \cdot y' = 0$$

such that $p(t,y) = h'_t$ and $q(t,y) = h'_y$ for some function $h = h(t,y)$.

Solution: $h(t,y) = C$

general solution
in implicit form

Ex:

$$\underbrace{t^2}_p + \underbrace{y^2}_{q} \cdot y' = 0$$

Work: $h'_t = t^2$
 $h'_y = y^2$

$$h = \frac{1}{3}t^3 + \frac{1}{3}y^3 + C$$

yes, the diff. equ.
is exact

and: the general solution
is given by

$$h = C$$

$$\frac{1}{3}t^3 + \frac{1}{3}y^3 + C_1 = C_2$$

$$\frac{1}{3}t^3 + \frac{1}{3}y^3 = C$$

$$\frac{1}{3}y^3 = C - \frac{1}{3}t^3 \Rightarrow y^3 = 3C - t^3 \Rightarrow y = \sqrt[3]{3C - t^3}$$

implicit solution

(explicit form)

Remarks:

i) $t^2 + y^2 \cdot y' = 0$ exact $\Rightarrow y = \underline{\underline{\sqrt[3]{K - t^3}}}$

$$y^2 \cdot y' = -t^2$$

Separable

$$y' = \underbrace{-t^2}_{f(t)} \cdot \underbrace{\frac{1}{y^2}}_{g(y)}$$

$$y^2 \cdot y' = -t^2$$

$$\int y^2 dy = \int -t^2 dt$$

$$\frac{1}{3}y^3 = -\frac{1}{3}t^3 + C \quad | \cdot 3$$

$$y^3 = -t^3 + 3C$$

$$y = \sqrt[3]{-t^3 + 3C} = \underline{\underline{\sqrt[3]{K - t^3}}}$$

ii) Exact: $h = \frac{1}{3}t^3 + \frac{1}{3}y^3 = C$

$t^2 + y^2 \cdot y' = 0$

$\underbrace{t^2}_{h'_t} + \underbrace{y^2 \cdot y'}_{h'_y} = 0$

implicit
derivation
 $\frac{d}{dt}(-)$

iii) Test for exactness:

$$p(t,y) + q(t,y) \cdot y' = 0 \text{ is exact}$$

Ex: $t^2 + y^2 \cdot y' = 0$

$$\underbrace{p'_y}_{=0} = \underbrace{q'_t}_{=0}$$

$$(t^2)'_y = 0 \quad (y^2)'_t = 0 \leftarrow \text{Yes, it is exact.}$$

Ex: $y' = \frac{3t^2 - y^2}{2ty}$ $\cdot 2ty$ not separable

$$2ty \cdot y' = 3t^2 - y^2$$

$$(y^2 - 3t^2) + \underbrace{2ty \cdot y'} = 0$$

p

q

$$h'_x + h'_y \cdot y' = 0$$

exact?

$$p + q \cdot y' = 0$$

$$\begin{cases} h'_t = y^2 - 3t^2 \\ h'_y = 2ty \end{cases}$$

Concl:

Yes, it is exact
and

$$h = y^2 t - t^3 + \underline{0} = C$$

$$ty^2 - t^3 = C$$

$$\frac{ty^2}{t} = \frac{C + t^3}{t}$$

$$y^2 = \frac{C + t^3}{t}$$

$$y = \pm \sqrt{\frac{C + t^3}{t}}$$

$$\begin{aligned} \textcircled{1} \quad h &= \int y^2 - 3t^2 dt \\ &= y^2 t - t^3 + C(y) \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad h'_y &= (y^2 t - t^3 + C(y))'_y \\ &= 2yt - 0 + C'(y) \\ &= \underline{2yt} + C'(y) = \underline{2ty} \end{aligned}$$

$$C'(y) = 0$$

$$C(y) = C$$

$$y' = \frac{3t^2 - y^2}{2ty}, \quad \underline{y(1) = 2} :$$

$$\underline{y(1) = 2} :$$

General solution

$$y = \pm \sqrt{\frac{C + t^3}{t}}$$

$$2 = \pm \sqrt{\frac{C + 1}{1}} = (+) \sqrt{C + 1}$$

$$4 = C + 1 \quad \underline{C = 3}$$

Particular solution

$$y = \sqrt{\frac{3 + t^3}{t}}$$

Plan

- 1 Linear first order differential equations
- 2 Superposition principle

① Linear first order differential equations

Defn: A first order differential equation is linear if it can be written

$$\boxed{y' + a(t) \cdot y = b(t)} \Leftrightarrow y' = \underbrace{b(t) - a(t)y}_{\substack{\uparrow \\ \text{linear in } y}}$$

Ex: $y' = y + t \Leftrightarrow \boxed{y' - y = t}$ $a(t) = -1$
 $b(t) = t$
is linear

$e^{2y} \cdot y' = te^t$ is not linear
 $y' = te^t \cdot e^{-2y}$

Solution method: Integrating factor

Ex: $y' - y = t$ 1. $u \leftarrow$ integrating factor
 choose $u = u(t)$ such that $\underline{u' = -u}$

$uy' - uy = ut$
 $\underline{uy' + u'y} = ut$
 $(uy)' = ut$
 $\int (uy)' dt = \int ut dt$
 $uy = \int t \cdot u(t) dt$
 $y = \frac{1}{u(t)} \int t \cdot u(t) dt \checkmark$

Formula: $u(t) = e^{\int a(t) dt}$
 $\Rightarrow u'(t) = e^{\int a(t) dt} \cdot a(t)$
 $= u(t) \cdot (-1)$
 $= -u(t)$

Ex: $a(t) = -1 \Rightarrow \int -1 dt = -t + C \Rightarrow$ Int. factor $u = e^{-t+C}$
 $u = \underline{e^{-t}}$

$$y' - y = t \cdot 1 \cdot e^{-t}$$

$$e^{-t} y' - e^{-t} y = t e^{-t}$$

$$(e^{-t} y)' = t e^{-t}$$

$$e^{-t} y = \int t e^{-t} dt$$

$$y = e^t \int t e^{-t} dt$$

$$= e^t (-e^{-t} \cdot t - \int -e^{-t} \cdot 1 dt)$$

$$= e^t (-t e^{-t} + (-e^{-t}) + C)$$

$$= e^t (-t e^{-t} - e^{-t} + C)$$

$$y = \underline{\underline{-t - 1 + C e^t}}$$

general
solution
(explicit form)

Integration factor:
 $\frac{1}{\int a(t) dt}$
 $u = e$

In general:

$$y' + a(t)y = b(t) \quad | \cdot u$$

$$u y' + u \cdot a(t)y = b(t) \cdot u(t)$$

$$(u \cdot y)' = b(t) \cdot u(t)$$

$$u \cdot y = \int b(t) \cdot u(t) dt$$

$$y = \frac{1}{u(t)} \int b(t) \cdot u(t) dt$$