

Plan

- 1 Second order differential equations
- 2 Linear second order differential equations

Review:

First order differential equations

- separable
- exact
- linear

$$y' = F(t, y)$$

① Second order differential equations

A differential equation that involves y'' , but no higher derivatives, is called a second order differential equation.

Ex:

$$y'' = 12t - 4 \quad | \int - dt$$

$$y' = \int 12t - 4 dt$$

$$= 12 \cdot \frac{t^2}{2} - 4t + C$$

$$y' = 6t^2 - 4t + C \quad | \int - dt$$

$$y = \int 6t^2 - 4t + C dt$$

$$= 6 \cdot \frac{t^3}{3} - 4 \frac{t^2}{2} + Ct + D$$

$$y = \underline{\underline{2t^3 - 2t^2 + Ct + D}}$$

general solution

depends on
two undetermined
coefficients

② Linear second order differential equations

Defn: A linear second order differential equation is a differentiation that can be written

$$y'' + a(t) \cdot y' + b(t) \cdot y = h(t)$$

We say that it is homogeneous if $h(t) = 0$, and it has constant coefficients if $a(t) = a$ and $b(t) = b$ are constants.

We are going to explain how to find the general solution in the case with constant coefficients.

(a) Solution method in the homogeneous case

Ex: $y'' - 3y' + 2y = 0$

Characteristic equation: $r^2 - 3r + 2 = 0$
 $(r-2)(r-1) = 0$
 $\underline{r=2}$ or $\underline{r=1}$

$y = \underline{C_1 \cdot e^{2t} + C_2 \cdot e^t}$
 (general solution)

$$r^2 + ar + b = 0 \quad \alpha \quad \beta$$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \pm \frac{\sqrt{4b - a^2}}{2} \cdot \sqrt{-1}$$

$$y'' + ay' + by = 0$$

Char. eqn: $r^2 + ar + b = 0$
 $r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$$\Delta = a^2 - 4b$$

$\Delta > 0$: two distinct roots $r_1 \neq r_2$
 $y = \underline{C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}}$

$\Delta = 0$: $r_1 = r_2 = -a/2$
 $y = \underline{C_1 e^{-a/2 \cdot t} + C_2 t e^{-a/2 \cdot t}}$

$\Delta < 0$: no real roots
 $y = e^{\alpha t} \cdot (C_1 \cos(\beta t) + C_2 \sin(\beta t))$

Ex: $y'' + 6y' + 9y = 0$

$$r^2 + 6r + 9 = 0$$

($\Delta = 0$) $r = \frac{-6 \pm \sqrt{36 - 4 \cdot 9}}{2}$

$$r_1 = r_2 = -3$$

$$y = \underline{\underline{C_1 e^{-3t} + C_2 t e^{-3t}}} = (C_1 + C_2 t) e^{-3t}$$

Ex: $y'' - 4y' + 5y = 0$

$$r^2 - 4r + 5 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4} \cdot \sqrt{-1}}{2}$$

($\Delta < 0$) $= 2 \pm 1 \cdot \sqrt{-1}$

$\alpha = 2$ $\beta = 1$

$$y = \underline{\underline{e^{2t} (C_1 \cos t + C_2 \sin t)}}$$

Explanation of the characteristic eqn:

Look for solutions
of the form $y = e^{rt}$

$$\left. \begin{aligned} y &= e^{rt} = 1 \cdot e^{rt} \\ y' &= e^{rt} \cdot r = r \cdot e^{rt} \\ y'' &= r(e^{rt} \cdot r) = r^2 \cdot e^{rt} \end{aligned} \right\}$$

$$\text{in } \boxed{y'' + ay' + by = 0}$$

$$r^2 e^{rt} + a(r e^{rt}) + b \cdot (1 \cdot e^{rt}) = 0$$

$$r^2 e^{rt} + a r e^{rt} + b e^{rt} = 0$$

$$(r^2 + ar + b) \cdot e^{rt} = 0$$

$$\underline{r^2 + ar + b = 0}$$

char. eqn.

Concl:

e^{rt} is a solution of

$$y'' + ay' + by = 0$$



$$r^2 + ar + b = 0$$

$$y' + ay = 0$$

$$u = e^{\int a dt} = e^{at+C}$$

int. factor: e^{at}

$$y' e^{at} + a y e^{at} = 0$$

$$(y \cdot e^{at})' = 0$$

$$y e^{at} = C$$

$$\underline{y = C \cdot e^{-at}}$$

(b) Solution method in the inhomogeneous case

In general: $y'' + ay' + by = h(t)$

Ex: $y'' - 3y' + 2y = e^{-t}$ inhomogeneous

Solution method: Superposition principle

General solution:

$$y = y_h + y_p$$

the general solution
of the homogeneous
equation

$$y'' + ay' + by = 0$$

a particular
solution of

$$y'' + ay' + by = h(t)$$

Ex: $y'' - 3y' + 2y = e^{-t}$

Superposition: $y = y_h + y_p = C_1 e^{2t} + C_2 e^t + \frac{1}{6} e^{-t}$

y_h : $y'' - 3y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$

$$r = 2, r = 1$$

$\Rightarrow$

$$y_h = C_1 e^{2t} + C_2 e^t$$

y_p : $y'' - 3y' + 2y = e^{-t}$

"Guess" $y = A \cdot e^{-t}$

$$y' = -A e^{-t}$$

$$y'' = A e^{-t}$$

$$(A e^{-t}) - 3(-A e^{-t}) + 2(A e^{-t}) = e^{-t}$$

$$(A + 3A + 2A) e^{-t} = e^{-t}$$

$$= 1$$

$$6A = 1 \Rightarrow A = \frac{1}{6} \Rightarrow y_p = \frac{1}{6} e^{-t}$$

Method of undetermined coeff:

How to "guess":

- should depend on parameters (A)
- should have the same form as $h(t), h'(t), h''(t)$
" e^{-t} " e^{-t} " e^{-t}

If it doesn't work,
try to multiply with t

Explanation: Superposition principle

$$y'' + a(t)y' + b(t)y = h(t)$$

linear second order differential equation

Differential operator

$$D = \frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t) \quad ; \quad D(y(t)) = y'' + a(t)y' + b(t)y$$

Ex: $y'' - 3y' + 2y = e^{-t}$
 $D(y) = e^{-t}$ where $D = \frac{d^2}{dt^2} - 3\frac{d}{dt} + 2$

$$D(t) = 0 - 3 \cdot 1 + 2t = -3 + 2t$$

$$D(e^{-t}) = (e^{-t})'' - 3 \cdot (e^{-t})' + 2 \cdot (e^{-t})$$

$$= e^{-t} - 3(-e^{-t}) + 2e^{-t} = 6e^{-t}$$

D is linear:
differential operator

$$\begin{aligned} \textcircled{1} \quad D(y_1 + y_2) &= D(y_1) + D(y_2) \\ \textcircled{2} \quad D(c \cdot y) &= c \cdot D(y) \end{aligned}$$

Applications:

i) $D(e^{-t}) = 6e^{-t} \Rightarrow D\left(\frac{1}{6}e^{-t}\right) = \frac{1}{6}D(e^{-t})$
 $= \frac{1}{6} \cdot 6e^{-t} = \underline{e^{-t}}$

ii) $y'' - 3y' + 2y = 0 \quad D = \frac{d^2}{dt^2} - 3\frac{d}{dt} + 2$

$$r^2 - 3r + 2 = 0$$

$$r = 2, r = 1 \Rightarrow \underline{e^{2t}, e^t \text{ are solutions}}$$

$y = \underline{C_1 e^{2t} + C_2 e^t}$ is the general solution

Linear property: $D(e^{2t}) = 0$
 $D(e^t) = 0$

$$\Rightarrow D(C_1 e^{2t} + C_2 e^t) = D(C_1 e^{2t}) + D(C_2 e^t) = C_1 D(e^{2t}) + C_2 D(e^t) = 0 + 0 = 0$$

iii) Superposition principle:

$$= y_h + y_p$$

general solution

$$y'' + a(t)y' + b(t)y = h(t)$$

$$\underline{D(y) = h(t)}$$

$$D = \frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t)$$

linear diff. operator

(a) Assume $y = y_h + y_p$, where

Then:

$$\begin{aligned} D(y) &= D(y_h + y_p) \\ &= D(y_h) + D(y_p) \\ &= 0 + h(t) = h(t) \end{aligned}$$

y_h : general solution of the homogeneous equation

$$D(y_h) = 0$$

y_p : particular solution

$$D(y_p) = h(t)$$

$\Rightarrow y$ is a solution

(b) Assume y is a solution, i.e. $D(y) = h(t)$

Then

$$\begin{aligned} D(y - y_p) &= D(y) - D(y_p) \\ &= h(t) - h(t) = 0 \end{aligned}$$

y_p : a particular solution,
i.e. $D(y_p) = h(t)$

$\Rightarrow y - y_p$ is a solution of the homogeneous eqn.

$$\Rightarrow y - y_p = y_h \Rightarrow y = y_h + y_p$$

Ex:

$$y' - y = e^{2t}$$

$$D(y) = e^{2t}$$

first order linear diff. eqn.

$$D = \frac{d}{dt} - 1$$

D has the linear property:

$$\begin{aligned} D(y_1 + y_2) &= (y_1 + y_2)' - 1(y_1 + y_2) \\ &= y_1' + y_2' - y_1 - y_2 \\ &= (y_1' - y_1) + (y_2' - y_2) \\ &= D(y_1) + D(y_2) \end{aligned}$$

Can use superposition

$$y = y_h + y_p = \underline{C e^t + e^{2t}}$$

y_h : $y' - y = 0$

$$r - 1 = 0 \quad r = 1$$

$$y_h = C \cdot e^t$$

y_p : $y' - y = e^{2t}$

$$y = A e^{2t} \quad A = 1$$

$$y' = 2A e^{2t} \quad y_p = 1 \cdot e^{2t}$$

$$2A e^{2t} - A e^{2t} = e^{2t}$$

$$\begin{aligned} D(e^{2t}) &= 2e^{2t} - e^{2t} \\ &= e^{2t} \\ y_p &= e^{2t} \end{aligned}$$

$$\begin{aligned} D(C \cdot y_1) &= (C y_1)' - 1(C y_1) \\ &= C y_1' - C \cdot 1 \cdot y_1 \\ &= C (y_1' - y_1) \\ &= C \cdot D(y_1) \end{aligned}$$

Summary: Linear differential eqn's

① $y' + a(t)y = b(t)$ first order linear

② $y'' + a(t)y' + b(t)y = h(t)$ second order linear

* Superposition always works : $y = y_h + y_p$

* Characteristic eqn: works if ① $a(t) = a$ constant
 ② $a(t) = a$ constant
 $b(t) = b$ constant

Plan

- 1 Equilibrium states and stability
- 2 Superposition principle

① Equilibrium states and stability:

Defn: A first order differential eqn. is called autonomous if it can be written $y' = F(y)$ ← no t in the diff. eqn.

Ex: $y' = 2y$
 $y' = 5y \cdot (1 - y/10)$

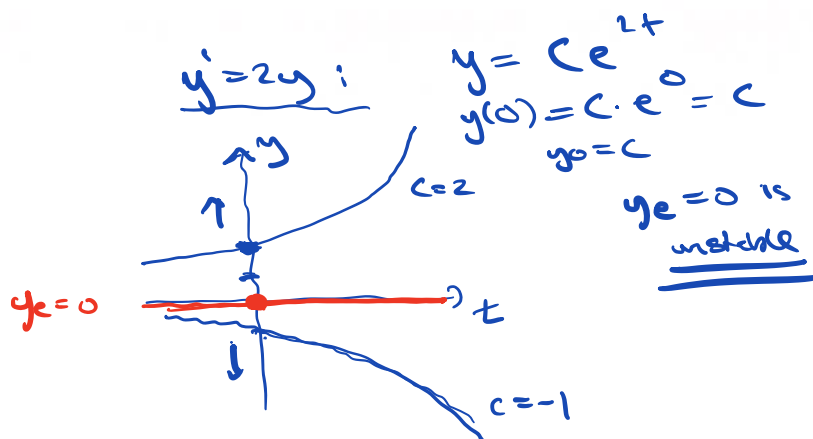
A equilibrium state of $y' = F(y)$ is value $y = y_e$ such that $F(y_e) = 0$

Ex: 1) $y' = 2y$

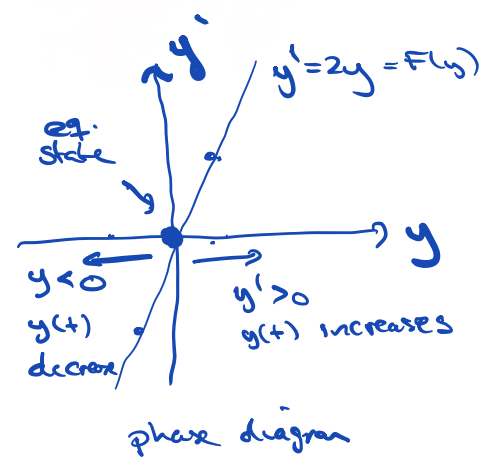
Eg. states: $2y = 0$
 $y = 0 \Rightarrow \underline{\underline{y_e = 0}}$

2) $y' = 5y(1 - y/10)$

Eg. states: $5y(1 - y/10) = 0$
 $y = 0$ or $y = 10$
 $5y = 0 \quad 1 - y/10 = 0$
 $\parallel$
 $y_e = 0, y_e = 10$



If $y_0 = 0$: $y = 0$



Defn. An eq. state $y = y_e$ is called

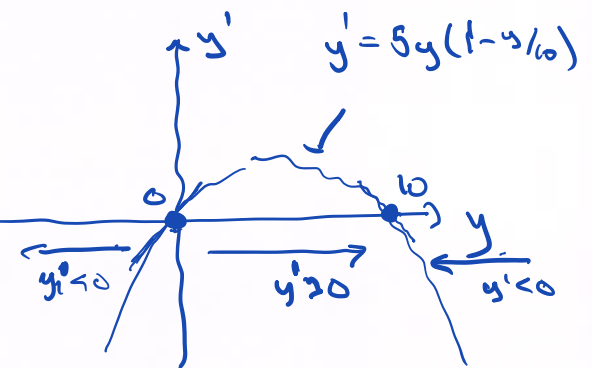
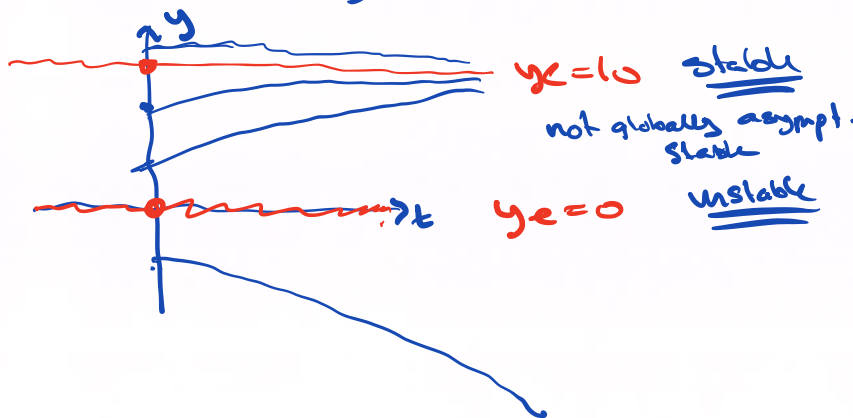
i) stable if y_0 close y_e means that $y(t)$ moves towards y_e — — — — — moves

ii) unstable if y_0 — — — — — further away from y_e

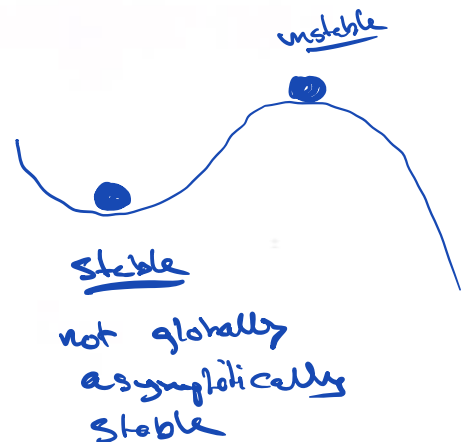
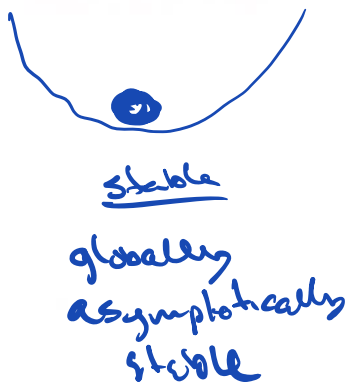
iii) globally asymptotically stable if $y(t)$ moves towards y_e no matter which starting point y_0 you consider

Ex: $y' = 5y(1 - y/10)$
 $= 5y - \frac{1}{2}y^2$

$y_e = 0$ | $y_e = 10$



phase diagram



Stability Thm:

If $y = y_e$ is an eq. state of $y' = F(y)$, then we have:

$F'(y_e) > 0 \Rightarrow y_e$ is unstable

$F'(y_e) < 0 \Rightarrow y_e$ is stable