Plan

1. Systems of linear first order differential equations
2. Homogeneous case
3. Inhomogeneous case

Review:

* Linear second order differential equations $y'' + ay' + by = f(t)$
* Superposition principle
* Equilibrium states and stability of $y' = F(y)$ (first order autonomous diff. eqns)

Plenary Session 4:
Monday 22/11 at 17 in A1-040
Problems from Lecture 10-13

TA sessions:
Monday 15/11
Monday 29/11 ?

① Systems of first order linear differential equations

Ex: $y_1' = y_1 + 4y_2$, $y_2' = 4y_1 + y_2$

Solutions: $y_1(t)$, $y_2(t)$

$y_1' - y_1 = 4y_2 \leftarrow$ coupled differential equations

Ex: $y_1' = 2y_1$, $y_2' = y_1 - y_2$ $\leftarrow$ decoupled : $y_1' - 2y_1 = 0$

$y_1 = y_1^h = \underline{C_1 e^{2t}}$

y_1^h: $r - 2 = 0$, $r = 2$

$y_2' + y_2 = y_1 = C_1 e^{2t}$

$y_2 = y_2^h + y_2^p = \underline{C_2 \cdot e^{-t} + \frac{C_1}{3} e^{2t}}$

y_2^h: $r + 1 = 0$, $r = -1 \Rightarrow y_2^h = C_2 e^{-t}$

y_2^p: $y_2' + y_2 = C_1 e^{2t}$

Try $y = Ae^{2t}$, $y' = 2Ae^{2t}$ $\Big\}$ $2Ae^{2t} + Ae^{2t} = C_1 e^{2t}$, $3Ae^{2t} = C_1 e^{2t}$

$3A = C_1 \Rightarrow A = C_1/3$

Conclusion:

$$y_1 = C_1 e^{2t}$$
$$y_2 = \frac{C_1}{3} e^{2t} + C_2 e^{-t}$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{2t} \\ \frac{C_1}{3} e^{2t} + C_2 e^{-t} \end{pmatrix}$$

$$= C_1 \cdot \begin{pmatrix} 1 \\ 1/3 \end{pmatrix} e^{2t} + C_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

In general:

System of first order differential equations

$$\boxed{\begin{aligned} y_1' &= F_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= F_2(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(t, y_1, y_2, \ldots, y_n) \end{aligned}}$$

It is called autonomous if F_i does not depend on t for $i = 1, 2, \ldots, n$.

It is called linear if $F_i(y_1, y_2, \ldots, y_n)$ are linear functions.

$$\boxed{\begin{aligned} y_1' &= a_{11} y_1 + a_{12} y_2 + \ldots + a_{1n} y_n + b_1 \\ y_2' &= a_{21} y_1 + a_{22} y_2 + \ldots + a_{2n} y_n + b_2 \\ &\vdots \\ y_n' &= a_{n1} y_1 + a_{n2} y_2 + \ldots + a_{nn} y_n + b_n \end{aligned}}$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad \mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}$$

Matrix form: $\boxed{\mathbf{y}' = A \cdot \mathbf{y} + \mathbf{b}}$

$\mathbf{b} = \mathbf{0}$: homogeneous case

$\mathbf{b} \neq \mathbf{0}$: inhomogeneous case

$$\underbrace{\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}}_{\mathbf{y}'} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\mathbf{y}} + \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\mathbf{b}}$$

$n \times n$ – matrix

Ex: $y_1' = y_1 + 4y_2$
$y_2' = 4y_1 + y_2$

Matrix form:
$\underline{y}' = A \cdot \underline{y}$, $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$

$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

② Homogeneous case: $\underline{y}' = A \cdot \underline{y}$, A $n \times n$-matrix

Ex: $\underline{y}' = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \underline{y}$ $\quad y_1' = y_1 + 4y_2$, $y_2' = 4y_1 + y_2$

Change of variables: P invertible matrix $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \iff \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Ex: $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$
$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$
$y_1 = z_1 - z_2$
$y_2 = z_1 + z_2$
$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$
$z_1 = \frac{1}{2}(y_1 + y_2)$
$z_2 = \frac{1}{2}(-y_1 + y_2)$

Transform the system $\underline{y}' = A\underline{y}$ into a system in $\underline{z}$:

$\underline{y} = P \cdot \underline{z}$: $\quad \underline{y}' = (P\underline{z})' = P \cdot \underline{z}'$
$\quad\quad \| \quad\quad\quad\quad \|$
$\quad A\underline{y} = A \cdot (P\underline{z})$

Transformed system: $P \cdot \underline{z}' = AP \cdot \underline{z} \quad | P^{-1} \cdot$
$\underline{z}' = P^{-1}AP \cdot \underline{z}$

If A is diagonalizable: $P^{-1}AP = D$ is diagonal
$\lambda_1, \lambda_2, \ldots, \lambda_n$: eigenvalues $\quad D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$
$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$: eigenvectors $\quad P = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n)$
$(A \cdot \underline{v}_i = \lambda_i \underline{v}_i$ for $i = 1, 2, \ldots, n)$

$\underline{z}' = (P^{-1}AP) \cdot \underline{z}$
$\underline{z}' = D \cdot \underline{z}$

$\begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

$\boxed{\begin{matrix} z_1' = \lambda_1 z_1 \\ z_2' = \lambda_2 z_2 \\ \vdots \\ z_n' = \lambda_n z_n \end{matrix}}$

$z_i' - \lambda_i z_i = 0$

decoupled
$z_1 = C_1 e^{\lambda_1 t}$
$z_2 = C_2 e^{\lambda_2 t}$
$\vdots$
$z_n = C_n e^{\lambda_n t}$

$$\underline{z} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix} \Rightarrow \underline{y} = P\underline{z} = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}) \cdot \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= \underline{v_1} \cdot c_1 e^{\lambda_1 t} + \underline{v_2} c_2 e^{\lambda_2 t} + \dots + \underline{v_n} c_n e^{\lambda_n t}$$

Result:

If A is a diagonalizable $n \times n$ matrix, with n eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicity) and n linearly independent eigenvectors $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ such that $A \cdot \underline{v_i} = \lambda_i \underline{v_i}$, then the system $\underline{y}' = A \cdot \underline{y}$ of first order differential eqns has general solution

$$\underline{y} = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \cdot \underline{v_2} e^{\lambda_2 t} + \dots + c_n \cdot \underline{v_n} e^{\lambda_n t}$$

Ex: $\underline{y}' = A\underline{y}$ with $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$

symmetric $\Rightarrow$ diagonalizable

$$\begin{aligned} y_1' &= y_1 + 4y_2 \\ y_2' &= 4y_1 + y_2 \end{aligned}$$

Eigenvalues: $\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 2\lambda + (-15) = 0$$
$$(\lambda - 5)(\lambda + 3) = 0$$
$$\lambda_1 = 5, \quad \lambda_2 = -3$$

Eigenvectors:

$\lambda = 5$: $\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \to \begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix}$ $\quad \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -3$: $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \to \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix}$ $\quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

General solution: $\underline{y} = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$

Ex: $y'' - 4y' + 3y = 0$

$$\left.\begin{array}{l} y_1 = y \\ y_2 = y' \end{array}\right\} \quad \begin{array}{l} y_1' = y' = y_2 \\ y_2' = y'' = 4y' - 3y = 4y_2 - 3y_1 \end{array}$$

$$\begin{array}{l} y_1' = y_2 \\ y_2' = -3y_1 + 4y_2 \end{array} \qquad \mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \mathbf{y}$$

Solve it as a system

$A = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$: $\begin{vmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{vmatrix} = 0$ $\quad \boxed{\lambda^2 - 4\lambda + 3 = 0}$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda_1 = 3, \lambda_2 = 1$

E_3: $\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \to \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

E_1: $\begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution: $\mathbf{y} = C_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + C_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} + C_2 e^t \\ 3C_1 e^{3t} + C_2 e^t \end{pmatrix}$$

Compare with: $y'' - 4y' + 3y = 0$

Char eqn: $\boxed{r^2 - 4r + 3 = 0}$

$y = C_1 e^{3t} + C_2 e^t$

③ Inhomogeneous case

Ex:
$$y_1' = y_1 + 4y_2 + 2$$
$$y_2' = 4y_1 + y_2 + 5$$

$$\boldsymbol{y}' = \underbrace{\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}}_{A} \boldsymbol{y} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Solution method: $\boldsymbol{y}' = A \cdot \boldsymbol{y} + \underline{b}$

(1) Equilibrium states: $A \cdot \boldsymbol{y} + \underline{b} = \underline{0}$ ← $\boldsymbol{y}' = \underline{0}$

$A \cdot \boldsymbol{y} = -\underline{b}$ linear system

Ex: $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 4 & 1 & -5 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & -15 & 3 \end{array}\right) \qquad \begin{aligned} y_1 + 4y_2 &= -2 \\ -15y_2 &= 3 \end{aligned}$$

$$y_2 = \frac{3}{-15} = -\underline{1/5} \qquad y_1 = -2 - 4(-1/5) = 4/5 - 10/5 = \underline{-6/5}$$

$$\boldsymbol{y}_e = \begin{pmatrix} -6/5 \\ -1/5 \end{pmatrix} \quad \text{eq. state}$$

(2) Assume that $\boxed{\boldsymbol{y}' = A\boldsymbol{y} + \underline{b}}$ has eq. state $\boldsymbol{y}_e$

Change of variables: $\underline{z} = \boldsymbol{y} - \boldsymbol{y}_e$ or $\boldsymbol{y} = \underline{z} + \boldsymbol{y}_e \Rightarrow$

$$\boldsymbol{y}' = (\underline{z} + \boldsymbol{y}_e)' = \underline{z}' + \underline{0} = \underline{z}'$$
$$\boldsymbol{y}' = A\boldsymbol{y} + \underline{b} = A(\underline{z} + \boldsymbol{y}_e) + \underline{b} = A\underline{z} + \underbrace{A\boldsymbol{y}_e + \underline{b}}_{=\underline{0}} = A\underline{z}$$

New system in $\underline{z}$: $\underline{z}' = A\underline{z}$ (homogeneous)

Solution: $\boldsymbol{y} = \underline{z} + \boldsymbol{y}_e$ ← eq. state

$\underline{z}$ ← Solution of $\underline{z}' = A\underline{z}$

Ex: $\underline{y}' = \underbrace{\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}}_{A} \underline{y} + \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{\underline{b}}$

i) $\underline{y_e}$: Solve $A \cdot \underline{y} + \underline{b} = 0 \Rightarrow \underline{y_e} = \begin{pmatrix} -6/5 \\ -1/5 \end{pmatrix}$

ii) $\underline{z}$: Solve $\underline{z}' = A\underline{z}$

$$\underline{z} = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$$

From Part 1:

$\lambda_1 = -3, \; \lambda_2 = 5$

$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution:

$$\underline{y} = \underline{z} + \underline{y_e} = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \begin{pmatrix} -6/5 \\ -1/5 \end{pmatrix}$$

Plan

1 Equilibrium states and stability

2 Examples

(1) Equilibrium states and stability

Consider $\underline{y}' = A \cdot \underline{y} + \underline{b}$ (system of linear first order differential eqns)

Defn: An equilibrium state is a constant vector $\underline{y}$ such that $A \cdot \underline{y} + \underline{b} = \underline{0}$. We write $\underline{y}_e$ for the equilibrium states.

Note: $|A| \neq 0 \Rightarrow$ there is a unique eq. state $\underline{y}_e$

$$\boxed{\begin{aligned} A\underline{y} + \underline{b} &= \underline{0} \\ A\underline{y} &= -\underline{b} \quad | \; A^{-1}\cdot \\ \underline{y}_e &= A^{-1}(-\underline{b}) = -A^{-1}\underline{b} \end{aligned}}$$

Defn: $\underline{y}_e$ is called stable if $\underline{y}_0$ close to $\underline{y}_e$ implies that $\underline{y}(t) \to \underline{y}_e$ as $t \to \infty$.

(1) If A has n negative eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n < 0$ then the eq. state $\underline{y}_e$ of $\underline{y}' = A\underline{y} + \underline{b}$ is globally asymptotically stable

(2) If at least one eigenvalue $\lambda > 0$ then the eq. state $\underline{y}_e$ is unstable

Ex: $\underline{y}' = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \underline{y} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$\underline{y}_e = \begin{pmatrix} -6/5 \\ -1/5 \end{pmatrix}$

unstable

Eigenvalues:

$\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 2\lambda - 15 = 0$

$\lambda = -3, \; \lambda = 5$

$\underline{y} = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + \begin{pmatrix} -6/5 \\ -1/5 \end{pmatrix}$

c_1, c_2 depends on $\underline{y}(0)$

Ex: $y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} y + \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad y(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: Eigenvalues $\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda_1 = 3, \lambda_2 = 1$

Eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eq. state: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} y + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} y = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

$\left(\begin{array}{cc|c} 2 & 1 & -3 \\ 1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & -3 \end{array}\right) \begin{matrix} \\ -2 \end{matrix}$

$\rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & -3 \end{array}\right)$ $\quad y_1 = -2, \; y_2 = 1$ $\quad y_e = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

General solution: $\boxed{y = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

Initial condition: $y(0) = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^0 + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^0 + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$c_1 - c_2 - 2 = -1$

$c_1 + c_2 + 1 = 1$

$\boxed{\begin{aligned} c_1 - c_2 &= 1 \\ c_1 + c_2 &= 0 \end{aligned}}$ $\quad c_1 = 1/2, \; c_2 = -1/2$

$$y = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$t \to \infty$: $y_1(t) \to \infty$, $y_2(t) \to \infty$ unstable

Eq. states / stability of $y'' + ay' + by = c$

= Eq. states / stability of the corresponding system.

Ex: $\boxed{y'' + 3y' + 2y = 4}$ ← Constant solutions:

$\boxed{\begin{array}{l} y = A \\ y' = 0 \\ y'' = 0 \end{array}}$

$2y = 4$

$y = 2$

$0 + 3 \cdot 0 + 2 \cdot A = 4$

$A = 2$

Solution: $y = y_h + y_p$

$= C_1 e^{-2t} + C_2 e^{-t} + 2$

$\downarrow 0 \quad \downarrow 0$ as $t \to \infty$

As system:

$\left.\begin{array}{l} y_1 = y \\ y_2 = y' \end{array}\right\}$ $\begin{array}{l} y_1' = y_2 \\ y_2' = 4 - 2y_1 - 3y_2 \end{array}$

$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Eq. states: $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & -4 \end{array}\right)$ $\quad y_2 = 0$, $\quad -2y_1 = -4 \quad y_1 = 2$

$\mathbf{y}_e = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad \begin{array}{l} y = 2 \\ y' = 0 \end{array}$

Eigenvalues: $\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0$

$\lambda = -2, \ \lambda = -1$

$\Longrightarrow$ neg. eigenvalues

$\mathbf{y}_e$ is globally asymptotically stable

Ex: $y''' - 2y' + y = 4 \iff y''' = 2y' - y + 4$

$y = y_h + y_p$

$y_h: \ r^3 - 2r + 1 = 0$

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \end{aligned}\right\} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y''' = -y_1 + 2y_2 + 4 \end{aligned}$$

$$\mathbf{y}' = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}}_{= A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}}_{= \mathbf{b}}$$

Eigenvalues:

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & 2 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 2) - 1(1) = 0$$

$-\lambda^3 + 2\lambda - 1 = 0 \quad | \cdot (-1)$

$\lambda^3 - 2\lambda + 1 = 0$

$(\lambda - 1) \cdot (\lambda^2 + \lambda - 1) = 0$

$\lambda = 1$ or $\lambda^2 + \lambda - 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1+4}}{2}$

$\underline{\lambda_1 = 1} \quad \underline{\lambda_2 = \frac{-1+\sqrt{5}}{2}} \quad \underline{\lambda_3 = \frac{-1-\sqrt{5}}{2}}$

You can use similar methods for solving these kinds of differential equations