Plan

1 Orthogonal diagonalization
2 Definiteness of quadratic forms

Review: $\boxed{A \cdot \underline{v} = \lambda \cdot \underline{v}}$

A $n \times n$ matrix

i) Eigenvalues: $|A - \lambda I| = 0$

$\boxed{(\lambda - \lambda_i)^{m_i} \text{ factor in } |A - \lambda I| \Leftrightarrow \lambda = \lambda_i \text{ has multiplicity } m_i}$

ii) Eigenvectors: $E_\lambda = \text{Null}(A - \lambda I)$ for each eigenvalue λ

$\boxed{1 \le \dim E_{\lambda_i} \le m_i}$

iii) Diagonalizable:

A diagonalizable with $P^{-1}AP = D$ if and only if

(i) there are n eigenvalues, $\lambda_1, \ldots, \lambda_n$, counted with multiplicity

and

(ii) there are n linearly independent eigenvectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$

$$\boxed{D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \qquad P = (\underline{v_1} | \underline{v_2} | \cdots | \underline{v_n})}$$

Result: A symmetric $\Rightarrow$ A diagonalizable

1 Orthogonal diagonalization

Defn.: A matrix P is called orthogonal if $P^T = P^{-1}$

$P = (\underline{v_1} | \underline{v_2} | \cdots | \underline{v_n})$: $P^T \cdot P = I$ ← condition for P to be orthogonal

$$\begin{pmatrix} \underline{v_1} \\ \underline{v_2} \\ \vdots \\ \underline{v_n} \end{pmatrix} \cdot (\underline{v_1} | \underline{v_2} | \cdots | \underline{v_n}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$\Updownarrow$

$$\boxed{\begin{aligned} &\|\underline{v}_i\| = 1 \\ &\underline{v}_i \perp \underline{v}_j \ (\forall\, i \ne j) \end{aligned}} \Leftrightarrow \left\{ \begin{aligned} &\underline{v_i} \cdot \underline{v_i} = 1 \\ &\underline{v_i} \cdot \underline{v_j} = 0 \quad \text{if } i \ne j \end{aligned} \right\}$$

orthonormal set of vectors

An <u>orthogonal diagonalization</u> of A is given by an orthogonal matrix P and a diagonal matrix D such that

$$P^T A P = D$$

To find an orthogonal diagonalization, we need to find an orthonormal ~~base~~ base for E_λ for each eigenvalue λ.

<u>Result</u>: A has an orthogonal diagonalization $\Longleftrightarrow$ A is symmetric

$$P^TAP = D \Rightarrow \cancel{P(P^TAP)P^T} = PDP^T \Rightarrow A^T = (PDP^T)^T$$
$$A = PDP^T \qquad = (P^T)^T D^T P^T$$
$$= PDP^T = A$$

<u>A is symmetric</u>

<u>Ex</u>:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

symmetric

<u>Eigenvalues</u>:

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \qquad +(2-\lambda)\cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\underline{\lambda = 2}$ or $\lambda^2 - 4\lambda + 3 = 0$

$(\lambda-3)(\lambda-1) = 0$

$\underline{\lambda=3}, \underline{\lambda=1}$

<u>Eigenvectors</u>:

$\underline{\lambda=2}$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{\lambda=3}$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$: $\begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda=1}$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$: $\begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$\underline{v_1}\cdot\underline{v_2} = 0 \qquad \underline{v_1}\cdot\underline{v_3} = 0 \qquad \underline{v_2}\cdot\underline{v_3} = 0$

$\underline{v_1}\cdot\underline{v_1} = 1 \qquad \underline{v_2}\cdot\underline{v_2} = 2 \qquad \underline{v_3}\cdot\underline{v_3} = 2$

$\qquad\qquad\qquad \|\underline{v_2}\| = \sqrt{2} \qquad \|\underline{v_3}\| = \sqrt{2}$

$$\left.\begin{aligned} \underline{v_1}' &= \underline{v_1} = (0,1,0) \\ \underline{v_2}' &= \tfrac{1}{\sqrt{2}}(1,0,1) \\ \underline{v_3}' &= \tfrac{1}{\sqrt{2}}(-1,0,1) \end{aligned}\right.$$

(2) Definiteness of quadratic forms

Defn: A function $f(x_1, x_2, \ldots, x_n)$ in n variables is called a quadratic form if it is a polynomial where all terms have degree two.

Ex: $f(x) = ax^2$ $\quad f(x,y) = ax^2 + bxy + cy^2$

$$f(x_1, x_2, \ldots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + \ldots + c_{1n}x_1x_n + c_{22}x_2^2 + c_{23}x_2x_3 + \ldots$$

Note: Any quadratic form has $f(0, 0, \ldots, 0) = 0$ and $(0, 0, \ldots, 0)$ is a stationary point.

Fact: Any quadratic form can be written in matrix form as

$$f(\underline{x}) = \underline{x}^T \cdot A \cdot \underline{x}$$

for a unique symmetric matrix A.

Ex: $f(x_1, x_2, x_3) = x_1^2 + \underline{4x_1x_2} - x_2^2 + 2x_3^2 = \underline{x}^T A \underline{x}$, $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$: $(x_1 \; x_2 \; x_3) \cdot \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Fact: Bijective (1-1) correspondence:

$f(\underline{x})$ quadratic form in n variables $\Longleftrightarrow$ A symmetric $n \times n$ matrix with $f(\underline{x}) = \underline{x}^T A \underline{x}$

Defn: Let $f(x_1, \dots, x_n)$ be a quadratic form

defn = definite

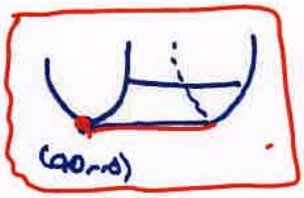
We say that:

i) f positive defn.
if $f(x_1, \dots, x_n) > 0$
for all $(x_1, \dots, x_n) \neq (0, 0, \dots, 0)$

$\Rightarrow$ $(0,0,\dots,0)$ unique global min for f

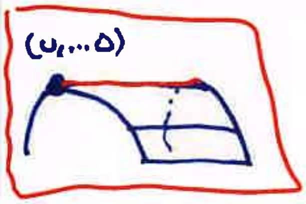
ii) f negative defn:
if $f(x_1, \dots, x_n) < 0$
for all $(x_1, \dots, x_n) \neq (0, 0, \dots, 0)$

$\Rightarrow$ $(0,0,\dots,0)$ unique global max for f



iii) f positive ~~positive~~ semidefn.
if $f(x_1, \dots, x_n) \geq 0$ for all $(x_1, \dots, x_n)$

$\Rightarrow$ $(0,0,\dots,0)$ global min for f



iv) f negative semidefn
if $f(x_1, \dots, x_n) \leq 0$
for all $(x_1, \dots, x_n)$

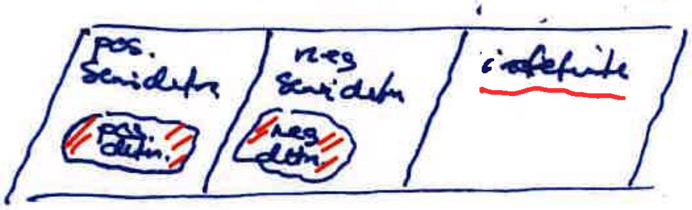
$\Rightarrow$ $(0,0,\dots,0)$ global max for f

v) f indefinite if
$f(x_1, \dots, x_n)$ can take both positive and negative values

$\Updownarrow$

neither positive nor negative semidefn.

$\Leftarrow$ $(0,0,\dots,0)$ saddle pt. for f



Ex:

$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2$
pos. defn.

$f(x_1, x_2) = x_1^2 - x_2^2$
indefinite

$f(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 7x_3^2$
?

Methods for determining definiteness of quadratic forms

(A) Eigenvalues: $f(\underline{x}) = \underline{x}^T A \underline{x}$

A symmetric matrix $\rightsquigarrow$ Eigenvalues of A: $\lambda_1, \lambda_2, \dots, \lambda_n$

Result:

$$\boxed{\begin{array}{lll} f \text{ positive defn.} & \Leftrightarrow & \lambda_1, \lambda_2, \dots, \lambda_n > 0 \\ \text{— '' — semidefn.} & \Leftrightarrow & \lambda_1, \lambda_2, \dots, \lambda_n \geq 0 \\ f \text{ negative defn.} & \Leftrightarrow & \lambda_1, \dots, \lambda_n < 0 \\ \text{— '' — semidefn.} & \Leftrightarrow & \lambda_1, \lambda_2, \dots, \lambda_n \leq 0 \\ f \text{ indefinite} & \Leftrightarrow & \text{there are both positive and negative eigenvalues} \end{array}}$$

Ex: $f(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 2xz$

$= \underline{x}^T A \underline{x}, \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$\lambda = 2, \lambda = 3, \lambda = 1$

f is positive definite

(0,0,0) is global min for f

$f(\underline{x}) = \underline{x}^T A \underline{x} =$

$= (P\underline{u})^T A (P\underline{u})$

$= \underline{u}^T \underbrace{P^T A P}\, \underline{u} = \underline{u}^T D \underline{u}$

$= 2u_1^2 + 3u_2^2 + u_3^2$

Orthogonal diagonalization of A:

$P^T A P = D$

$P = \begin{pmatrix} 0 & +1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Define $\underline{x} = P\underline{u}$ or $\boxed{P^T \underline{x} = \underline{u}}$

$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ +1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$

$u_1 = y$

$u_2 = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z$

$u_3 = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z$

Note: Since A is symmetric, there is an orthogonal diagonalization $P^T A P = D$. When we choose new coordinates $\underline{u}$ such that $\underline{x} = P\underline{u}$ (or $\underline{u} = P^T \cdot \underline{x}$) then

$$\boxed{f(\underline{x}) = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \dots + \lambda_n u_n^2}$$

(B) Principal minors:

Defn: A principal minor of order r is a minor $M_{I,J}$ such that $I = J$. We write Δ_r for any of the principal minors of order r.

The leading principal minor of order r is

$D_r = M_{12\dots r, 12\dots r}$ (choose the first r rows/cols)

Ex: $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$$\Delta_1 \begin{cases} M_{1,1} = 2 \leftarrow D_1 \\ M_{2,2} = 2 \\ M_{3,3} = 2 \end{cases}$$

$$\Delta_2 \begin{cases} M_{12,12} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \leftarrow D_2 \\ M_{13,13} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \\ M_{23,23} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \end{cases}$$

$$\Delta_3 \left\{ |A| = M_{123,123} = 2(4-1) = 6 \leftarrow D_3 \right.$$

Result: A $n \times n$ symm. matrix

A positive defn. $\iff$ $D_1, D_2, \ldots, D_n > 0$

A negative defn. $\iff$ $D_1 < 0$, $D_2 > 0$, $D_3 < 0, \ldots$

$[(-1)^i D_i > 0$ for $i = 1, 2, \ldots, n]$

(check all leading principal minors)

Result:

A positive semidefn. $\iff$ $\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$ ✓

A negative semidefn. $\iff$ $\Delta_1 \leq 0$, $\Delta_2 \geq 0, \ldots$ ✓

$[(-1)^i \Delta_i \geq 0$ for $i = 1, 2, \ldots, n]$

(check all principal minors)

All other cases: indefinite

Typical examples: D_2 or $\Delta_2 < 0 \Rightarrow$ indefinite; $D_1 > 0$, $D_3 < 0 \Rightarrow$ indefinite

Ex: $f(x,y,z) = -x^2 + 4xy - 5y^2 - 2z^2$.

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$D_1 = -1 < 0$

$D_2 = 5 - 4 = 1 > 0$

$D_3 = -2 \cdot D_2 = -2 \cdot 1 = -2 < 0$

f negative defn.

$f(x,y,z) = 2x^2 + 8y^2 - 8xy + z^2$

$$A = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$D_1 = 2 > 0$

$D_2 = 16 - 16 = 0$

$D_3 = 1 \cdot D_2 = 1 \cdot 0 = 0$

pos. ~~defn~~. semidefn.?

or

indefinite?

$\Delta_1 = 2, 8, 1 \geq 0$

$\Delta_2 = 0, 8, 2 \geq 0$

$\Delta_3 = 0 \geq 0$

$\Downarrow$

positive semidefinite

Note: Why $D_1 < 0, D_2 > 0, D_3 < 0, \ldots \iff$ negative definite

Ex: $f(x,y,z) = -x^2 - 2y^2 - z^2$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -1$ — (A) neg. def.

$D_1 = -1 < 0$
$D_2 = (-1)(-2) = 2 > 0$
$D_3 = (-1)(-2)(-1) = -2 < 0$ — (B) neg. def.

(C) Reduced rank criterion

Ex: $f(x,y,z,w) = 2x^2 + y^2 + 2z^2 + w^2 + 2xz - 2yw$

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$D_1 = \underline{2}$
$D_2 = \underline{2}$
$D_3 = 1 \cdot (4-1) = \underline{3}$
$D_4 = \underline{0}$

$\Delta_1 = 2, 1, 2, 1$
$\Delta_2 = 2, 2, 2, \ldots$
$\Delta_3 = 3, \ldots$
$\Delta_4 = 0$

$R(4) = -R(2)$

Reduced rank criterion (RRC):

A symmetric $n \times n$ matrix: If $rk(A) = r < n$ (reduced rank), then we have:

$D_1, D_2, \ldots, D_r > 0 \implies A$ positive semidefinite

$(-1)^i D_i > 0$ for $i = 1, \ldots, r \implies A$ negative semidefinite

$\Updownarrow$

$D_1 < 0, D_2 > 0, \ldots, (-1)^r D_r > 0$

In the ex:

$D_4 = 0 \implies rk(A) < 4$: $rk(A) = \underline{\underline{3}}$ since $D_3 \neq 0$

$$\left.\begin{array}{l} D_1 = 2 \\ D_2 = 2 \\ D_3 = 3 \end{array}\right\}$$ RRC: A is positive semidef.

Ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$

$D_1 = 2$
$D_2 = 5$
$D_3 = 0$

$R(3) = R(1) + R(2)$

$\Updownarrow$

$f = 2x^2 + 3y^2 + 7z^2 + 2xy + 6xz + 8yz$

RRC: $\text{rk } A = 2$

$D_1, D_2 > 0$

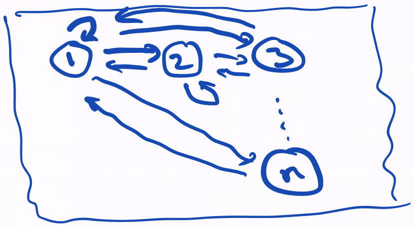
$\Downarrow$

A pos. semidef.

Plan

1. Non-negative matrices and Markov chains
2. Projections and orthonormal bases

① Markov chains:



n states

State vector:

$\underline{x} = (x_1, x_2, x_3, \dots, x_n)$

s.t. $x_i \geq 0$

$x_1 + x_2 + x_3 + \dots + x_n = 1$

"share of population in state i" $= x_i$

Transition matrix:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

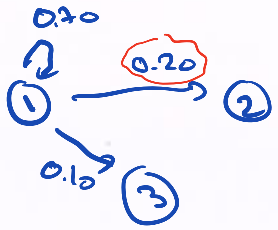
$a_{ij} \geq 0$

the column sums $= 1$

$n \times n$ matrix

$a_{ij} =$ share of population in state j that transitions to state i in one time period

Ex:



$$A = \begin{pmatrix} 0.70 = a_{11} & * & * \\ 0.20 = a_{21} & * & * \\ 0.10 = a_{31} & * & * \end{pmatrix}$$

Equations:

$\underline{v}_0$, $\underline{v}_{t+1} = A \cdot \underline{v}_t$

initial state vector

$\underline{v}_1 = A \cdot \underline{v}_0$

$\underline{v}_2 = A \cdot \underline{v}_1 = A^2 \underline{v}_0$

$\vdots$

$\underline{v}_m = A^m \cdot \underline{v}_0$

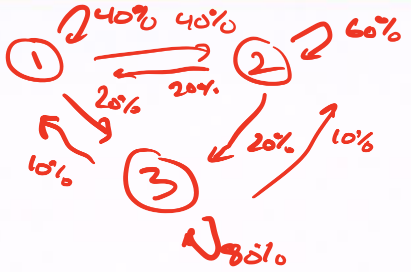
Defn: A Markov chain is called regular if there is an integer $m \geq 1$ such that you can get from any state to any other state in exactly m steps.

$A^m > 0$ for some int. $m \geq 1$

If $A > 0$ (i.e. $a_{ij} > 0$ for all i, j) then it is regular. $(m = 1)$

Theorem: If A is the transition matrix of a regular Markov chain, then we have: ① $\lambda = 1$ is an eigenvalue of multiplicity $m = 1$, all other eigenvalues satisfy $|\lambda| < 1$ ② There is a unique eigenvector $\underline{v}$ in E_1 that is a state vector, and this is the equilibrium state, i.e. $\lim_{m \to \infty} A^m \cdot \underline{v}_0 = \underline{v}$

Ex:



$$A=\begin{pmatrix} 0.40 & 0.20 & 0.10 \\ 0.40 & 0.60 & 0.10 \\ 0.20 & 0.20 & 0.80 \end{pmatrix} \qquad \underline{v}_0=\begin{pmatrix} 0.35 \\ 0.30 \\ 0.35 \end{pmatrix}$$

$$\underline{v} = \lim_{m\to\infty} A^m \cdot \underline{v}_0 = \underline{\underline{(3/16 \;\; 5/16 \;\; 8/16)}}$$

- regular Markov chain since $\underline{A>0}$:

$\underline{\underline{\lambda=1}}$:

$$\begin{pmatrix} -0.6 & 0.2 & 0.1 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.2 \end{pmatrix} \to \begin{pmatrix} 0.2 & 0.2 & -0.2 \\ 0.4 & -0.4 & 0.1 \\ -0.6 & 0.2 & 0.1 \end{pmatrix} \begin{matrix} \\ -2 \\ 3 \end{matrix}$$

$$\to \begin{pmatrix} 0.2 & 0.2 & -0.2 \\ 0 & -0.8 & 0.5 \\ 0 & 0.8 & -0.5 \end{pmatrix} \begin{matrix} \cdot 10 \\ \cdot 10 \\ \cdot 10 \end{matrix} \to \begin{pmatrix} 2 & 2 & -2 \\ 0 & -8 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$2x+2y-2z=0$
$-8y+5z=0$
z free

$y = 5z/8$

$2x = -2(5z/8)+2z = -\frac{10}{8}z + \frac{16}{8}z = \frac{6}{8}z = \frac{3}{4}z$

$\Rightarrow x = 3/8 \cdot z$

$\underline{E_1}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{8}z \\ \frac{5}{8}z \\ z \end{pmatrix} = \frac{z}{8}\begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} \Rightarrow \underline{v}_1 = (3,5,8)$ is a basis of E_1.

$3+5+8 = \underline{16}$

Equilibrium state:

$$\underline{v} = \frac{1}{16}(3,5,8) = \underline{\underline{(3/16, 5/16, 8/16)}}$$

② Projections and orthonormal bases:

Ex: $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ symmetric

Eigenvalues: $\lambda_1 = \lambda_2 = +1$, $\lambda_3 = 4$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$\lambda = 1$: $(m=2)$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x + y + z = 0 \Rightarrow x = -y - z$

y, z free

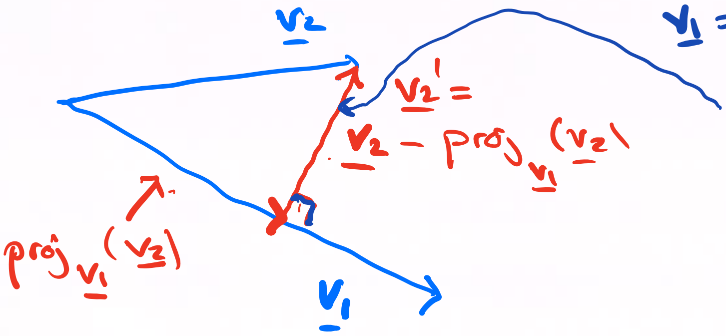
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

base of E_1

$v_1 \cdot v_1 = 2 \qquad v_2 \cdot v_2 = 2$

$v_1 \cdot v_2 = 1 + 0 + 0 = 1 \neq 0$



$v_1 = (-1, 1, 0)$

$v_2' = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0)$

$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$$\text{proj}_{v_1}(v_2) = \left(\frac{v_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$$

Projection formula:
See [E] Section 2.5

$v_1 \cdot v_2' = \frac{1}{2} \cdot (1 + (-1) + 0) = 0$

$$\boxed{w_1 = \frac{1}{\sqrt{2}}(-1, 1, 0), \quad w_2 = \frac{1}{\sqrt{6}}(-1, -1, 2)}$$

orthonormal base of E_1.