

Plan

- 1 Kuhn-Tucker problems
- 2 Second order conditions
- 3 Non-degenerate constraint qualification

Review: Lagrange problem $\max/\min f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Necessary conditions: $L = f(z) - \lambda_1 (f_1(z) - a_1) - \dots - \lambda_m (g_m(z) - a_m)$

i) Lagrange conditions Foc + c : $L'_{x_i} = 0$ $g_i(x) = a_i$

- ii) Admin. pts where NDCG fails

SOC: $(\underline{x}^*; \underline{\lambda}^*)$ satisfy FOC+CC, and $h(\underline{z}; \underline{\lambda}^*)$ convex $\Rightarrow \underline{x}^*$ min
 —||— concave $\Rightarrow \underline{x}^*$ max

EVT: f cont. on compact set $D \rightarrow f$ attains max/min on D

① Kuhn-Tucker problems = constrained optimization problem with closed inequality ($\leq, \geq$) constraints

Defn: A Kuhn-Tucker problem in standard form:

$$\max f(\underline{x}) \quad \text{subject to} \quad \begin{cases} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$
$$\text{Find } \max f(x, y, z) = x^2 + y^2 + z^2 \quad \text{when } xyz \geq 1 \quad | \cdot (-1)$$


 $\min f(x, y, z) = x^2 + y^2 + z^2$
 w.l.m.
 $-xyz \leq -1$ ← std. form
 $x + 2y - z \leq 6$

 $\max_{\partial D} f(x, y, z) = -x^2 - y^2 - z^2$

Ex: $\max f(x,y,z) = x^2 y^2 z^2$ wh $x^2 + y^2 + z^2 + x^2 y^2 z^2 \leq 4$

std.
form

$$L = x^2 y^2 z^2 - \lambda (x^2 + y^2 + z^2 + x^2 y^2 z^2 - 4)$$

Foc: $L'_x = 2xy^2z^2 - \lambda(2x + 2xy^2z^2) = 0$
 $L'_y = x^2 \cdot 2y \cdot z^2 - \lambda(2y + x^2 \cdot 2yz^2) = 0$
 $L'_z = x^2 y^2 \cdot 2z - \lambda(2z + x^2 y^2 \cdot 2z) = 0$

C: $x^2 + y^2 + z^2 + x^2 y^2 z^2 \leq 4$

CSC: $\lambda \geq 0$
 $\lambda \cdot (x^2 + y^2 + z^2 + x^2 y^2 z^2 - 4) = 0$

Complementary
slackness
conditions

(ND)

(D)

C: $x^2 + y^2 + z^2 + x^2 y^2 z^2 < 4$ | $x^2 + y^2 + z^2 + x^2 y^2 z^2 = 4$

CSC: $\lambda = 0$

Foc: $2xy^2z^2 = 0$
 $2x^2yz^2 = 0$
 $2x^2y^2z = 0$

$\Downarrow$

$x=0$ or $y=0$ or $z=0$

Many pts, all with $f=0$

Foc: $2x(y^2z^2 - \lambda(1 + y^2z^2)) = 0$
 $2y(x^2z^2 - \lambda(1 + x^2z^2)) = 0$
 $2z(x^2y^2 - \lambda(1 + x^2y^2)) = 0$

$x=0$ or $\lambda = \frac{y^2z^2}{1+y^2z^2}$

$y=0$ or $\lambda = \frac{x^2z^2}{1+x^2z^2}$

$z=0$ or $\lambda = \frac{x^2y^2}{1+x^2y^2}$

i) $x=0$ or $y=0$ or $z=0$: all pts have $f=0$

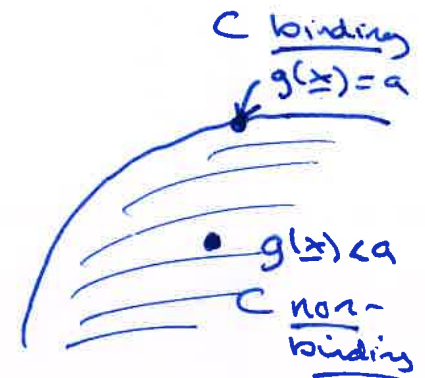
ii) $\lambda = \frac{y^2z^2}{1+y^2z^2} = \frac{x^2z^2}{1+x^2z^2} = \frac{x^2y^2}{1+x^2y^2}$

$y^2z^2(1+x^2z^2) = (1+y^2z^2)x^2z^2$
 $y^2z^2 + x^2y^2z^4 = x^2z^2 + x^2y^2z^4$
 $z^2(y^2 - x^2) = 0 \Rightarrow y^2 = x^2$

$y^2z^2 = \lambda \cdot (1 + y^2z^2)$
 $\lambda = \frac{y^2z^2}{1+y^2z^2}$

$A=0$ if $g(x) < a$
 $\lambda \geq 0$ if $g(x) = a$

Kuhn-Tucker
conditions: Foc + C + CSC



$\lambda \geq 0$ and
 $\lambda(g(x) - a) = 0$

$\Uparrow$

$A=0$ if $g(x) < a$
 $\lambda \geq 0$ if $g(x) = a$

$$\Rightarrow x^2 = y^2 = z^2 \quad \lambda = \frac{y^2 z^2}{1 + y^2 z^2} = \frac{x^2 \cdot x^2}{1 + x^2 \cdot x^2}$$

$$\underline{C:} \quad x^2 + y^2 + z^2 + x^2 y^2 z^2 = 4$$

$$3x^2 + x^6 = 4$$

$$x^6 + 3x^2 - 4 = 0$$

$$\underline{u=x^2:} \quad u^3 + 3u - 4 = 0 \quad \leftarrow u=1 \text{ is a solution}$$

$$(u-1)(u^2 + u + 4) = 0$$

$$\underline{u=1} \text{ or } u^2 + u + 4 = 0$$

$$\underline{x^2=1}$$

$$u = \frac{-1 \pm \sqrt{1-4 \cdot 4}}{2}$$

no solutions

$$\underline{\text{Candidate pts:}} \quad \underline{x^2 = y^2 = z^2 = 1, \lambda = 1/2}$$

$$\Rightarrow (\pm 1, \pm 1, \pm 1; 1/2) \quad f=1$$

Sol. of FOC + C + CSC

Evt:

f cont., D closed and bounded $\Rightarrow$ there is a max
 yes $\checkmark$ yes $?$
 $(\leq, \geq)$

$$\underline{C:} \quad \underline{x^2 + y^2 + z^2 + x^2 y^2 z^2 \leq 4}$$

D is bounded $\Rightarrow$ there is a max

$$x^2 \leq 4: \quad -2 \leq x \leq 2$$

$$y^2 \leq 4: \quad -2 \leq y \leq 2$$

$$z^2 \leq 4: \quad -2 \leq z \leq 2$$

$f(\pm 1, \pm 1, \pm 1) = 1$ is best candidate for max

among ordinary
 candidate pts
 (FOC + C + CSC)

NDCQ: Are there any adn. pts where
 NDCQ fails?

② NDCQ in the Kuhn-Tucker case

C: $g(x) \leq a$

" "
 $x^2 + y^2 + z^2 + x^2 y^2 z^2 \leq 4$

B: $x^2 + y^2 + z^2 + x^2 y^2 z^2 = 4$

~~NDCQ:~~
 $2x + 2y + 2z$

NB:

NDCQ (B):

$\text{rk} (g'_x \ g'_y \ g'_z) = 1$

$\text{rk} (2x + 2xy^2z^2 \ 2y + 2yx^2z^2 \ 2z + 2zx^2y^2) = 1$

NDCQ fails: $2x(1+y^2z^2) = 0 \quad 2y(1+x^2z^2) = 0 \quad 2z(1+x^2y^2) = 0$
 $x=0 \quad y=0 \quad z=0$

$(0,0,0)$ is not adm. with $x^2 + y^2 + z^2 + x^2 y^2 z^2 = 4$

NDCQ (NB):

no condition

Concl: In the example, there are no adm. pts where NDCQ fails.

||

Concl for the problem:

$f_{\max} = \underline{1}$ at $(x,y,z) = (\pm 1, \pm 1, \pm 1)$ with $\underline{\lambda = 1/2}$

NDCQ for a general Kuhn-Tucker case:

$\left. \begin{array}{l} g_1(x) \leq a_1 \\ g_2(x) \leq a_2 \\ \vdots \\ g_m(x) \leq a_m \end{array} \right\}$

Let x^* be an adm pt (satisfying all constraints) and let J be the matrix obtained from

$\begin{pmatrix} \partial g_1 / \partial x_1 & \partial g_1 / \partial x_2 & \dots & \partial g_1 / \partial x_n \\ \vdots & \vdots & & \vdots \\ \partial g_m / \partial x_1 & \partial g_m / \partial x_2 & \dots & \partial g_m / \partial x_n \end{pmatrix}$

by keeping rows that correspond to binding constraints.

NDCQ:

$\text{rk } J$ is maximal

Ex: NDCQ, KT problem with several constraints

$$\max -2x^2 - y^2 - 3z^2 \quad \text{when} \quad \begin{cases} -x + y - 2z \leq -3 \\ -x - y \leq -3 \end{cases}$$

C:

$$\begin{aligned} -x + y - 2z &\leq -3 \\ -x - y &\leq -3 \end{aligned}$$

full Jacobian:

$$\begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}$$

Cases:

i) B-B:

$$\begin{aligned} -x + y - 2z &= -3 \\ -x - y &= -3 \end{aligned}$$

$$\text{rk} \begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix} = 2$$

$M = 1 - (-1) = 2 \neq 0$

(ok)

ii) B-NB:

$$\begin{aligned} -x + y - 2z &= -3 \\ -x - y &< -3 \end{aligned}$$

$$\text{rk} \begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix} = 1$$

(ok)

iii) NB-B:

$$\begin{aligned} -x + y - 2z &< -3 \\ -x - y &= -3 \end{aligned}$$

$$\text{rk} \begin{pmatrix} -1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix} = 1$$

(ok)

iv) NB-NB:

$$\begin{aligned} -x + y - 2z &< -3 \\ -x - y &< -3 \end{aligned}$$

no condition

(ok)

Conclusion: No adn. pts where NDCQ fails.

In general: $\max f(\underline{x})$ where $\left\{ \begin{array}{l} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{array} \right.$ $\leftarrow$ std. form

$$L = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \lambda_2 (g_2(\underline{x}) - a_2) - \dots - \lambda_m (g_m(\underline{x}) - a_m)$$

FOC: $\begin{array}{l} L'_{x_1} = 0 \\ L'_{x_2} = 0 \\ \vdots \\ L'_{x_n} = 0 \end{array}$

C: $\begin{array}{l} g_1(\underline{x}) \leq a_1 \\ g_2(\underline{x}) \leq a_2 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{array}$

CSC: $\begin{array}{ll} \lambda_1 \geq 0 & \lambda_1 (g_1(\underline{x}) - a_1) = 0 \\ \lambda_2 \geq 0 & \lambda_2 (g_2(\underline{x}) - a_2) = 0 \\ \vdots & \vdots \\ \lambda_m \geq 0 & \lambda_m (g_m(\underline{x}) - a_m) = 0 \end{array}$

Thm (Necessary condition for Kuhn-Tucker problems in std form)

If $\underline{x}^*$ is a maxim pt in a Kuhn-Tucker problem in std. form, and NDCQ is satisfied at $\underline{x}^*$, then there are Lagrange multipliers $\underline{\lambda}^*$ such that $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC + C + CSC.

③ SOC: (in the Kuhn-Tucker case)

Thm:

If $(\underline{x}^*, \underline{\lambda}^*)$ satisfies FOC+C+CS in a Kuhn-Tucker problem in std. form, and

$$h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*) \text{ is concave}$$

then $\underline{x}^*$ is maximum point.

Ex: $\max x^2 y^2 z^2$ s.t. $x^2 + y^2 + z^2 + x^2 y^2 z^2 \leq 4$

Candidate: $(\pm 1, \pm 1, \pm 1; 1/2) \leftarrow$ satisfies FOC+C+CS

$$\begin{aligned} h(x, y, z) &= L(x, y, z, 1/2) = x^2 y^2 z^2 - \frac{1}{2}(x^2 + y^2 + z^2 + x^2 y^2 z^2 - 4) \\ &= \underline{-\frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 + \frac{1}{2}x^2 y^2 z^2 + 2} \end{aligned}$$

$$H(h) = \begin{pmatrix} -1+y^2z^2 & 0 & 0 \\ 0 & -1+x^2z^2 & 0 \\ 0 & 0 & -1+x^2y^2 \end{pmatrix} = \begin{pmatrix} y^2z^2-1 & 0 & 0 \\ 0 & x^2z^2-1 & 0 \\ 0 & 0 & x^2y^2-1 \end{pmatrix}$$

h is not concave.

No conclusion from SOC.

$$D_1 = y^2 z^2 - 1$$

can be both pos. / neg.

Plan

1 Envelope theorem for constrained optimization problems

Ex: $\max f(x, y, z, w) = xw - yz$ wh $\begin{cases} x^2 + y^2 = 16 \\ 4z^2 + 9w^2 = 36 \end{cases}$
 Lagrange problem (from Lecture 7, Part 1)

Concl: $f_{\max} = 12$ at $(x, y, z, w) = (0, 4, -3, 0)$ with $\lambda_1 = 3/8$
 $(0, -4, 3, 0)$ $\lambda_2 = 1/6$

What if: i) $4z^2 + 9w^2 = 35$
 ii) $f = xw - 2yz$

What happens with the max. value?

i) Consider: $\max f = xw - yz$ wh $\begin{cases} x^2 + y^2 = 16 \\ 4z^2 + 9w^2 = t \end{cases}$ $t = \text{parameter}$

Know: $f^*(t)$ max. value fn.

$\begin{cases} x^*(t) \\ y^*(t) \\ z^*(t) \\ w^*(t) \end{cases}$ max pt.

$\begin{cases} \lambda_1^*(t) \\ \lambda_2^*(t) \end{cases}$ Lagrange mult. at max pt.

$$f^*(36) = 12$$

$$x^*(36) = 0$$

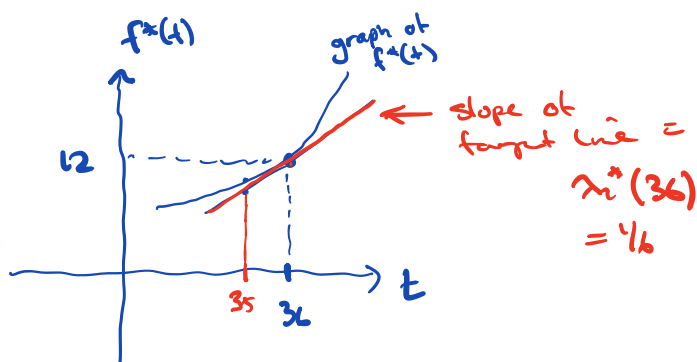
$$y^*(36) = 4 \text{ or } -4$$

$$z^*(36) = -3 \text{ or } 3$$

$$w^*(36) = 0$$

$$\lambda_1^*(36) = 3/8$$

$$\lambda_2^*(36) = 1/6$$



Envelope Thm:

$$\frac{df^*(t)}{dt} = L_t(\underline{x}^*(t); \underline{\lambda}^*(t))$$

We want to compute the right hand side

Estimate max. value when $t = 35$:

$$\begin{aligned} f^*(35) &\approx f^*(36) + \Delta t \cdot \frac{df^*(t)}{dt}(36) \\ &= 12 + (-1) \cdot 1/6 \\ &= 12 - 1/6 \approx 11.83 \end{aligned}$$

$$\begin{aligned} L &= xw - yz - \lambda_1(x^2 + y^2 - 16) - \lambda_2(4z^2 + 9w^2 - t) \\ &= \dots + \lambda_2 t \end{aligned}$$

$$L_t = \lambda_2 = 0 \quad L_t(\underline{x}^*(t); \underline{\lambda}^*(t)) = \lambda_2^*(t)$$

$$\text{At } t = 36: \quad \frac{df^*(t)}{dt} = \lambda_2^*(36) = 1/6$$

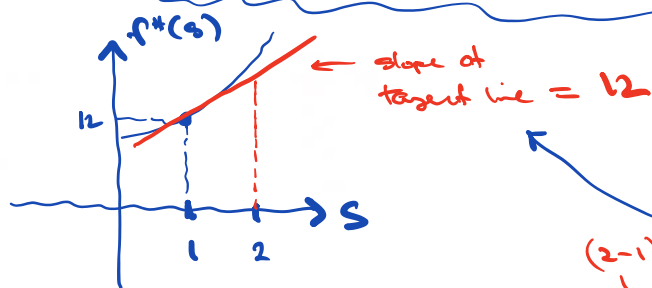
ii) $\max f = xw - s \cdot yz$ with $\begin{cases} x^2 + y^2 = 16 \\ 4x^2 + 9w^2 = 36 \end{cases}$ $s = \text{parameter}$

$s=1$:

$$f^*(1) = 12$$

$$x^*(1) = 0 \quad y^*(1) = 4 \text{ or } -4 \quad z^*(1) = -3 \text{ or } 3 \quad w^*(1) = 0$$

$$\lambda^*(1) = 3/8 \quad \lambda_2^*(1) = 1/6$$



Estimate: $f^*(2) \approx f^*(1) + \Delta s \cdot \frac{df^*(s)}{ds}$

$$= 12 + 1 \cdot 12$$

$$= 24$$

Using envelope thm:

$$L = xw - syz - \lambda_1(x^2 + y^2 - 16) - \lambda_2(4x^2 + 9w^2 - 36)$$

$$\frac{df^*(s)}{ds} = L'_s(x^*(s), z^*(s))$$

$$L'_s = -yz \Rightarrow \frac{df^*(s)}{ds} = -y^*(s)z^*(s)$$

$$\textcircled{s=1} = -y^*(1)z^*(1)$$

$$= -4 \cdot (-3) = 12$$

$$-(-4) \cdot 3 = 12$$

Note:

$$\lambda_i = \frac{df^*(a_i)}{da_i}$$

"marginal change in max/min value when you change a_i , the constant in the i th constraint"

max/min $f(x)$ with $\begin{cases} g_1(x) = a_1 \\ \vdots \\ g_m(x) = a_m \end{cases}$
general Lagrange problem