

Plan

- 1 Quadratic functions and optimization
- 2 Minimum variance portfolio

Part 2: at 12-13 today
(Zoom) (not 11-12)

Review:

Kuhn-Tucker problem
in std. form:

$$\max f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$

Candidate pts:

- i) $(\underline{x}^*; \underline{\lambda}^*)$ solves FOC + CSC
- ii) $\underline{x}^*$ adm. pt. where NDCQ fails

$$\begin{array}{lll} \lambda'_i = 0 & g_i(\underline{x}) \leq a_i & \lambda_i \geq 0, \lambda_i (g_i(\underline{x}) - a_i) = 0 \\ \text{FOC} & c & \text{CSC} \end{array}$$

Env. Thm (for Lagrange / Kuhn-Tucker problems)

$$\frac{df^*(a)}{da} = \lambda'_a(\underline{x}^*(a); \underline{\lambda}^*(a))$$

① Quadratic functions = polynomial fn. of degree 2

Ex: $f(x_1, x_2, x_3) = \underbrace{4x_1^2 - 4x_1x_2 + 2x_2^2 + 4x_1x_3 + 2x_3^2}_{\text{quadratic form}} \underbrace{- 4x_1 - 4x_3}_{\text{linear form}} \underbrace{+ 3}_{\text{constant}}$

$$f(\underline{x}) = \underline{x}^T A \underline{x} + B \underline{x} + C$$

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$B = (-4 \quad 0 \quad -4)$$

$$C = 3$$

Result: $f(\underline{x}) = \underline{x}^T A \underline{x} + B \underline{x} + C$

$$i) \quad f'(\underline{x}) = \begin{pmatrix} f'_1 \\ \vdots \\ f'_n \end{pmatrix} = 2A \underline{x} + B^T$$

$$ii) \quad H(\underline{x}) = 2A$$

Ex: $f(\underline{x}) = \underline{x}^T A \underline{x} + B \underline{x} + c$

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \quad B = (-4 \ 0 \ 4)$$

$$c = 3$$

Find max/min $f(\underline{x})$:

$$D_1 = 4 \quad D_2 = 4$$

$$D_3 = 2(-4) + 2 \cdot 4 = 0$$

i) Stationary pts: $f'(\underline{x}) = \underline{0}$ Foc

$$2A \cdot \underline{x} + B^T = \underline{0}$$

$$2A \underline{x} = -B^T$$

$$A \underline{x} = -\frac{1}{2} B^T$$

$$\left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ -2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 4 \end{array} \right) \xrightarrow{+1/2} \left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\underline{x} = 1 - z$$

$$4x = 4 - 4z$$

Stat pts:

$$(x, y, z) = (1 - z, 1 - z, z)$$

where z is free

$$4x = 2 + 2(1 - z) - 2z$$

$$y = 1 - z$$

z free

$$\left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

ii) Classify: $H(f)(\underline{x}^*) = 2A$ pos. semidef.

no concl.
from sec. der. test

iii) Convex/concave: $H(f) = 2A$ — 11 —

$\Rightarrow f$ convex $\Rightarrow (1 - z, 1 - z, z)$ are global min

Conclusion: $f_{\min} = f(0, 0, 1) = \underline{\underline{1}}$

$z = 1$ all global min pts have the same value (for all z)

Ex. max/min $f(\underline{x}) = x^2 + 2xy + 4y^2 + z^2$ wh $x+y+z=1$
 $= \underline{x}^T A \underline{x}$ $B \underline{x} = 1$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = (1 \ 1 \ 1)$$

$$h = \underline{x}^T A \underline{x} - \lambda \cdot (B \underline{x} - 1) \leftarrow \text{quadratic function in } \underline{x}$$

Foc: $L'(\underline{x}) = \begin{cases} 2A\underline{x} - \lambda(B^T) = \underline{0} \\ B\underline{x} = 1 \end{cases}$

C:

$$2A\underline{x} = \lambda \cdot B^T$$

$$A\underline{x} = \frac{\lambda}{2} \cdot B^T$$

$$\underline{x} = A^{-1} \left(\frac{\lambda}{2} B^T \right)$$

$$= \frac{\lambda}{2} (A^{-1} B^T)$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} d_1 = 1 \\ d_2 = 3 \\ d_3 = 1 \cdot 3 = 3 \end{matrix}$$

pos. defn.
(invertible)

$$B \underline{x} = 1$$

$$B \cdot \frac{\lambda}{2} A^{-1} B^T = 1$$

$$\frac{\lambda}{2} \cdot B A^{-1} B^T = 1$$

$$\lambda = \frac{2}{B A^{-1} B^T}$$

$$\underline{x} = \frac{1}{B A^{-1} B^T} \cdot A^{-1} B^T$$

compute: $A^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^T$

$$= \begin{pmatrix} 4/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B A^{-1} B^T = (1 \ 1 \ 1) \begin{pmatrix} 4/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2$$

$$\underline{x} = \frac{1}{2} \cdot \frac{1}{3} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \quad \lambda = 1 \rightarrow \text{and pts: } (x, y, z; \lambda) = (1/2, 0, 1/2; 1)$$

SOC: $L(\underline{x}; 1) = \underline{x}^T A \underline{x} - (B \underline{x} - 1) = h(\underline{x})$

$$H(h) = 2A \text{ pos. defn} \Rightarrow h \text{ convex} \Rightarrow f_{\min} = f(1/2, 0, 1/2) = 1/4 + 1/4 = 1/2$$

Ex: $\max/\min f(\underline{x}) = \underline{x}^T A \underline{x}$
 $= x^2 + 2xy + 4y^2 + z^2$

when $x^2 + y^2 + z^2 = 6$

$$\underline{x}^T \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{x} = 6$$

$$L = \underline{x}^T A \underline{x} - \lambda (\underline{x}^T I \underline{x} - 6)$$

$$\underline{x}^T \underline{x} = 6$$

Foc: $L'(\underline{x}) = \begin{cases} 2A\underline{x} - \lambda(2I\underline{x}) = \underline{0} \\ \underline{x}^T \cdot \underline{x} = 6 \end{cases}$

$$2A\underline{x} = 2\lambda I\underline{x} \quad | :2$$

$$A\underline{x} = \lambda I\underline{x} = \lambda \underline{x}$$

$$(A - \lambda I)\underline{x} = \underline{0} \iff |A - \lambda I| = 0 \quad \text{or } \underline{x} = \underline{0}$$

$\underline{x}$ is an
eigenvector
of A with
eigenvalue λ

$$\underline{0}^T \cdot \underline{0} = 0 \neq 6$$

not possible

$$\underline{x}^T \cdot \underline{x} = 6$$

$$\|\underline{x}\|^2 = 6$$

$$\|\underline{x}\| = \sqrt{6}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \cdot [(1-\lambda)(4-\lambda) - 1] = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda^2 - 5\lambda + 3 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 12}}{2}$$

$$= \frac{5 \pm \sqrt{13}}{2}$$

$$\lambda_1 = 1: \quad E_1 = \text{span}(\underline{v}_1)$$

two candidate pts

$$f = 6$$

$$\lambda_2 = \frac{5 + \sqrt{13}}{2}$$

≈ 4.60 two candidate pts

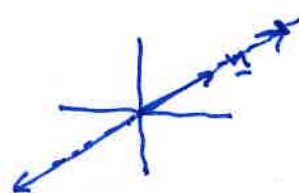
$\max \rightarrow f = 3(5 + \sqrt{13})$

$$\lambda_3 = \frac{5 - \sqrt{13}}{2}$$

≈ 0.70 two candidate pts

$\min \rightarrow f = 3(5 - \sqrt{13})$

$\underline{x}$ eigenvector of A in E_2 : $f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T \lambda \underline{x} = \lambda \underline{x}^T \underline{x} = 6\lambda$



② Minimum variance portfolio

Universe: n securities with returns $R_1, \dots, R_n$ with known

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

vector of expected returns

$$\mu_i = E(R_i)$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_n^2 \end{pmatrix}$$

Covariance matrix

$$\sigma_i^2 = \text{Var}(R_i)$$

$$\sigma_{ij} = \text{Cov}(R_i, R_j)$$

Portfolio:

$$\underline{w} = (w_1, w_2, \dots, w_n) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

w_i : weight of security #i in the portfolio

$$\boxed{w_1 + w_2 + \dots + w_n = 1} \Leftrightarrow \underline{e}^T \cdot \underline{w} = 1$$

$$\underline{R} = w_1 R_1 + \dots + w_n R_n$$

(return of the portfolio)

$$E(R) = w_1 \cdot E(R_1) + \dots + w_n \cdot E(R_n)$$

$$= (\mu_1 \mu_2 \dots \mu_n) \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \underline{\mu}^T \cdot \underline{w}$$

$$\text{Var}(R) = \text{Cov}(R, R) = \text{Cov}(w_1 R_1 + \dots + w_n R_n, w_1 R_1 + \dots + w_n R_n)$$

$$= w_1^2 \cdot \sigma_1^2 + w_1 w_2 \cdot \sigma_{12} + \dots + w_n^2 \sigma_n^2$$

$$= \underline{w}^T \Sigma \underline{w} \quad \Leftrightarrow \sigma_R = \sqrt{\text{Var}(R)}$$

Note:

a) Σ is symmetric and positive semidefinite

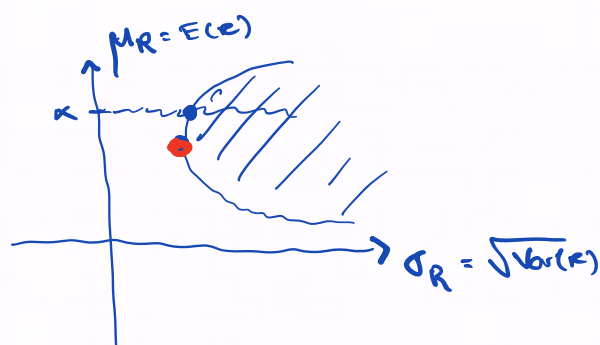
b) Reasonable assumption:

i) Σ positive defn.

ii) $\{\underline{\mu}, \underline{e}\}$ lin. independent $\left(\underline{e} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right) \rightarrow \text{if not, } \underline{\mu} = c \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix}$

Problem A: $\min_{\underline{w} \in \text{Var}(R)} \underline{w}^T \underline{\Sigma} \underline{w}$ when $\underline{e}^T \underline{w} = 1$

Problem B: $\min_{\underline{w} \in \text{Var}(R)} \underline{w}^T \underline{\Sigma} \underline{w}$ when $\begin{cases} \underline{e}^T \underline{w} = 1 \\ \underline{\mu}^T \underline{w} = \alpha \end{cases}$ for a given α



(A) $L = \underline{w}^T \underline{\Sigma} \underline{w} - \lambda (\underline{e}^T \underline{w} - 1)$

Foc: $L'(\underline{w}) = 2\underline{\Sigma} \underline{w} - \lambda \underline{e} = \underline{0}$

c: $\underline{e}^T \underline{w} = 1$

$2\underline{\Sigma} \underline{w} = \lambda \underline{e}$ Σ pos. defn. $\Rightarrow |\Sigma| \neq 0$

$\underline{\Sigma} \underline{w} = \frac{\lambda}{2} \underline{e}$

$\underline{w} = \underline{\Sigma}^{-1} \left(\frac{\lambda}{2} \underline{e} \right) = \frac{\lambda}{2} \underline{\Sigma}^{-1} \underline{e}$

$\underline{e}^T \left(\frac{\lambda}{2} \underline{\Sigma}^{-1} \underline{e} \right) = 1$

Cond. pt:

$\underline{w}^* = \frac{1}{\underline{e}^T \underline{\Sigma}^{-1} \underline{e}} \underline{\Sigma}^{-1} \underline{e} ; \lambda = \frac{2}{\underline{e}^T \underline{\Sigma}^{-1} \underline{e}}$

min. variance portfolio

$\lambda^* = \frac{2}{\underline{e}^T \underline{\Sigma}^{-1} \underline{e}}$

Sec: $h = L(\underline{w}; \lambda^*) = \underline{w}^T \underline{\Sigma} \underline{w} - \frac{2}{\underline{e}^T \underline{\Sigma}^{-1} \underline{e}} (\underline{e}^T \underline{w} - 1)$

$h(\underline{w}) = 2\underline{\Sigma}$ pos. defn $\Rightarrow h$ convex $\Rightarrow \underline{w}^*$ is min

(B) $L = \underline{w}^T \underline{\Sigma} \underline{w} - \lambda_1 (\underline{e}^T \underline{w} - 1) - \lambda_2 (\underline{\mu}^T \underline{w} - \alpha)$

Foc: $L'(\underline{w}) = 2\underline{\Sigma} \underline{w} - \lambda_1 \underline{e} - \lambda_2 \underline{\mu} = \underline{0}$

c:

$\left\{ \begin{array}{l} \underline{e}^T \underline{w} = 1 \\ \underline{\mu}^T \underline{w} = \alpha \end{array} \right.$

$\underline{e}^T \underline{w} = 1$
 $\underline{\mu}^T \underline{w} = \alpha$

$$2\underline{z} = \lambda_1 \underline{e} + \lambda_2 \underline{\mu}$$

$$\underline{z} = \underline{\Sigma}^{-1} \left(\frac{\lambda_1}{2} \underline{e} + \frac{\lambda_2}{2} \underline{\mu} \right) = \frac{\lambda_1}{2} \underline{\Sigma}^{-1} \underline{e} + \frac{\lambda_2}{2} \underline{\Sigma}^{-1} \underline{\mu}$$

$$\underline{e}^T \underline{z} = \frac{\lambda_1}{2} \underline{e}^T \underline{\Sigma}^{-1} \underline{e} + \frac{\lambda_2}{2} \underline{e}^T \underline{\Sigma}^{-1} \underline{\mu} = 1$$

$$\underline{\mu}^T \underline{z} = \frac{\lambda_1}{2} \underline{\mu}^T \underline{\Sigma}^{-1} \underline{e} + \frac{\lambda_2}{2} \underline{\mu}^T \underline{\Sigma}^{-1} \underline{\mu} = \alpha$$

$$\begin{matrix} \uparrow \\ (\underline{e}^T \underline{\Sigma}^{-1} \underline{e}) \cdot \lambda_1 + (\underline{e}^T \underline{\Sigma}^{-1} \underline{\mu}) \lambda_2 = 2 \end{matrix}$$

$$(\underline{\mu}^T \underline{\Sigma}^{-1} \underline{e}) \cdot \lambda_1 + (\underline{\mu}^T \underline{\Sigma}^{-1} \underline{\mu}) \lambda_2 = 2\alpha$$

Note:
$$\begin{pmatrix} \underline{e}^T \underline{\Sigma}^{-1} \underline{e} & \underline{e}^T \underline{\Sigma}^{-1} \underline{\mu} \\ \underline{\mu}^T \underline{\Sigma}^{-1} \underline{e} & \underline{\mu}^T \underline{\Sigma}^{-1} \underline{\mu} \end{pmatrix} = \underbrace{\begin{pmatrix} \underline{e} \\ \underline{\mu} \end{pmatrix}^T \underline{\Sigma}^{-1} \begin{pmatrix} \underline{e} \\ \underline{\mu} \end{pmatrix}}_{\text{pos. defn.}}$$

$$\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \underline{e}^T \underline{\Sigma}^{-1} \underline{e} & \underline{e}^T \underline{\Sigma}^{-1} \underline{\mu} \\ \underline{\mu}^T \underline{\Sigma}^{-1} \underline{e} & \underline{\mu}^T \underline{\Sigma}^{-1} \underline{\mu} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 2\alpha \end{pmatrix}$$

$$\underline{z} = \frac{\lambda_1}{2} \underline{\Sigma}^{-1} \underline{e} + \frac{\lambda_2}{2} \underline{\Sigma}^{-1} \underline{\mu}$$

One candidate pt. $\Rightarrow$ minimum
loc

Explanation: Why is $\underbrace{\begin{pmatrix} \underline{e} \\ \underline{\mu} \end{pmatrix}^T \underline{\Sigma}^{-1} \begin{pmatrix} \underline{e} \\ \underline{\mu} \end{pmatrix}}_B$ pos. defn.

Note:

B is a 2×2 symmetric matrix

$\Rightarrow$ look at its quadratic form

$$q(x, y) = (x \ y) B \begin{pmatrix} x \\ y \end{pmatrix} = (x \underline{e}^T + y \underline{\mu}^T) \underline{\Sigma}^{-1} (x \underline{e} + y \underline{\mu})$$

- i) $q(x,y) \geq 0$ since Σ^{-1} is pos. defn. symmetric matrix
- Σ^{-1} is pos. defn. symmetric
- * $\Sigma^{-1} = \frac{1}{|\Sigma|} \text{adj}(\Sigma)$ is symmetric since Σ is symmetric
 - * Σ pos defn $\Rightarrow$ all eigenvalues of Σ are positive $(\lambda_1, \lambda_2, \dots, \lambda_n)$ ($\lambda_i > 0$)
 - $\Rightarrow$ eigenvalues of Σ^{-1} are $1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_n > 0$
 - $\Rightarrow \Sigma^{-1}$ pos. defn.

$$q(x,y) = \underline{v}^T \Sigma^{-1} \underline{v} \geq 0 \quad \text{with } \underline{v} = x\underline{e} + y\underline{\mu}$$

(since Σ^{-1} pos. defn.)

- ii) $q(x,y) = 0 \Rightarrow \underline{v}^T \Sigma^{-1} \underline{v} = 0$ with $\underline{v} = x\underline{e} + y\underline{\mu}$
- $\Rightarrow \underline{v} = \underline{0}$ (since Σ^{-1} is pos. defn.)
- $\Rightarrow x\underline{e} + y\underline{\mu} = \underline{0}$
- $\Rightarrow x = y = 0$ (since $\{\underline{e}, \underline{\mu}\}$ are lin. independt)

Since $q(x,y) \geq 0$, and $q(x,y) = 0 \Rightarrow (x,y) = (0,0)$, it follows that q is positive definite.

$\Rightarrow$ Symmetric matrix of $q = B = \begin{pmatrix} \underline{e}^T \Sigma^{-1} \underline{e} & \underline{e}^T \Sigma^{-1} \underline{\mu} \\ \underline{\mu}^T \Sigma^{-1} \underline{e} & \underline{\mu}^T \Sigma^{-1} \underline{\mu} \end{pmatrix}$ pos. defn.

Plan

1 More examples: Constrained optimization

$$\max f(x,y) = \ln(x^2y) - x - y \quad \text{whn} \begin{cases} x+y \geq 4 & 1 \cdot (-1) \\ x \geq 1 & 1 \cdot (-1) \\ y \geq 1 & 1 \cdot (-1) \end{cases} \quad \text{KT}$$

std. $\downarrow$

$$\max f = \ln(x^2y) - x - y \quad \text{whn} \begin{cases} -x-y \leq -4 \\ -x \leq -1 \\ -y \leq -1 \end{cases} \quad \text{KT, std}$$

$$h = \ln(x^2y) - x - y - \lambda_1(-x-y+4) - \lambda_2(-x+1) - \lambda_3(-y+1)$$

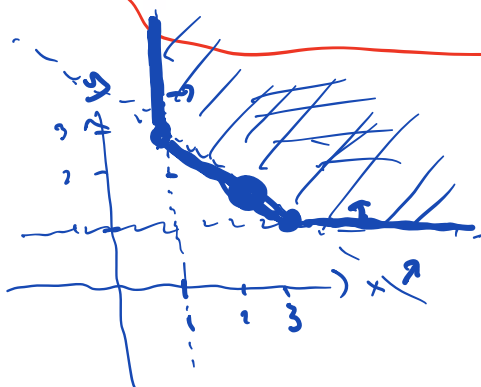
$$= 2\ln(x) + \ln(y) - x - y + \lambda_1(x+y-4) + \lambda_2(x-1) + \lambda_3(y-1)$$

Foc: $\lambda'_x = \frac{2}{x} - 1 + \lambda_1 + \lambda_2 = 0$

$\lambda'_y = \frac{1}{y} - 1 + \lambda_1 + \lambda_3 = 0$

C: $x+y \geq 4$
 $x \geq 1$
 $y \geq 1$

CSC: $\lambda_1 \geq 0 \quad \lambda_1(x+y-4) = 0$
 $\lambda_2 \geq 0 \quad \lambda_2(x-1) = 0$
 $\lambda_3 \geq 0 \quad \lambda_3(y-1) = 0$



Assume $x=1$:

$1 + \lambda_1 + \lambda_2 = 0$ impossible
 $\Rightarrow$ we must have: $x > 1, \lambda_2 = 0$

Assume $y=1$:

$0 + \lambda_1 + \lambda_3 = 0$
 $\Rightarrow \lambda_1 = 0, \lambda_3 = 0$

$\frac{2}{x} - 1 + 0 + 0 = 0 \Rightarrow x = 2$
 $x+y = 3$ impossible

Assume $x+y=4$:

$\frac{2}{x} - 1 + \lambda_1 = 0 \quad x = 1 - \frac{2}{x} = 1 - \frac{1}{y}$
 $\frac{1}{y} - 1 + \lambda_1 = 0 \quad \frac{2}{x} = \frac{1}{y}$

$2y = x$

$x+y = 2y+y = 4$
 $3y = 4$

$y = 4/3$
 $x = 8/3$

Kandidate pt:

$(x,y; \lambda_1, \lambda_2, \lambda_3) = (8/3, 4/3; 1/4, 0, 0)$ (OK)

SOC: $h = L(x, y; 14, 0, 0)$

$$= 2\ln x + \ln y - x - y + \frac{1}{4}(x+y-4)$$

$$H(h) = \begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix} \quad \begin{array}{l} \text{neg.} \\ \text{defn.} \end{array} \Rightarrow h \text{ concave} \Rightarrow (8/3, 4/3) \text{ is 'max'}$$

$$\underline{f_{\max} = f(8/3, 4/3) =}$$

$$= \ln\left(\left(\frac{8}{3}\right)^2 \left(\frac{4}{3}\right)\right) - \frac{8}{3} - \frac{4}{3}$$

$$= \underline{\underline{\ln(256/27) - 4}}$$