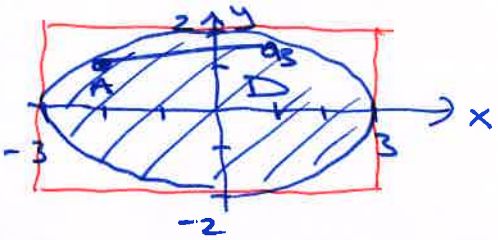
	Plan	
1	Key Problems:	7.1bd, 8.1ab, 9.1, 9.2b, 9.3, 9.4
2	Final Exams:	11/2019 Q4 ; 06/2018 Q4

① Key Problems

7.1 b) $D = \{(x,y) : 4x^2 + 9y^2 \leq 36\}$



$4x^2 + 9y^2 = 36 \quad | : 36$

$\frac{4x^2}{36} + \frac{9y^2}{36} = 1 \qquad \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$ ellipse, center $(0,0)$, half-axis $a = 3$, $b = 2$

D bounded since
$-3 \leq x \leq 3$
$-2 \leq y \leq 2$

D is convex since
$A, B \in D \Rightarrow [A,B] \subseteq D$

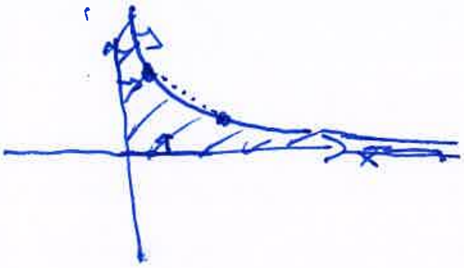
Without the figure:
$4x^2 + 9y^2 \leq 36$
$4x^2 \leq 36 \quad x^2 \leq 9$
$9y^2 \leq 36 \quad y^2 \leq 4$

$\boxed{\begin{array}{l} -3 \leq x \leq 3 \\ -2 \leq y \leq 2 \end{array}}$

D is bounded

Helpful to know: 2 var's
straight line $\Leftrightarrow$ linear eqn.
circle / ellipse $\Leftrightarrow$ $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$
hyperbola / parabola : $y = 1/x$, $y = x^2$

d) $D = \{(x,y) : 4xy \leq 1, \; x,y > 0\}$



$4xy = 1 \quad y = \frac{1}{4x} = \frac{1}{4} \cdot \frac{1}{x}$

D is not bounded, not closed, not convex

8.1 a) $\boxed{\max f = x - 2y + z \text{ when } x^2 + y^2 + z^2 \leq 3}$ KT prob. Std. form

$L = x - 2y + z - \lambda(x^2 + y^2 + z^2 - 3)$

FOC: $L'_x = 1 - \lambda \cdot 2x = 0$ C: $x^2+y^2+z^2 \leq 3$ CSC: $\lambda \geq 0$

$L'_y = -2 - \lambda \cdot 2y = 0$ $\lambda \cdot (x^2+y^2+z^2-3) = 0$

$L'_z = 1 - \lambda \cdot 2z = 0$

$\Downarrow$

$x = \frac{1}{2\lambda}$ $(\lambda \neq 0) \overset{CSC1}{\Rightarrow} \lambda > 0 \overset{CSC2}{\Rightarrow} x^2+y^2+z^2 = 3$

$y = \frac{-2}{2\lambda}$ $\frac{1^2}{(2\lambda)^2} + \frac{(-2)^2}{(2\lambda)^2} + \frac{1^2}{(2\lambda)^2} = 3 \quad | \cdot (2\lambda)^2$

$z = \frac{1}{2\lambda}$

$\frac{6}{3} = \frac{1 + 4 + 1}{3} = \frac{3 \cdot (2\lambda)^2}{3}$

$(x, y, z; \lambda) = \left(\frac{1}{2}\sqrt{2}, -\sqrt{2}, \frac{1}{2}\sqrt{2}; \frac{\sqrt{2}}{2}\right)$

$(2\lambda)^2 = 2$

$2\lambda = \pm\sqrt{2}$ $\lambda = \pm \frac{\sqrt{2}}{2} \overset{CSC'}{=} \frac{\sqrt{2}}{2}$

$\frac{1}{\lambda} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Conclusion:

$x^2+y^2+z^2 \leq 3$ compact set $\overset{EVT}{\Rightarrow}$ there is a max $\Rightarrow$ List of candidate pts:

① $\left(\frac{1}{2}\sqrt{2}, -\sqrt{2}, \frac{1}{2}\sqrt{2}; \frac{\sqrt{2}}{2}\right)$ $f = 3\sqrt{2}$

$-\sqrt{3} \leq x, y, z \leq \sqrt{3}$

② Adm pts. where NDCQ fails: none

$\Downarrow$

Check: $x^2+y^2+z^2 = 3$: $\text{rk}(2x \; 2y \; 2z) = 1$ fails $\Rightarrow (0,0,0) \Rightarrow$ not possible.

$x^2+y^2+z^2 < 3$: no cond.

$f_{max} = 3\sqrt{2}$

b) $\max f = xz + yw$ when $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$ KT-prob std. form

$$L = xz + yw - \lambda_1(x^2+y^2-1) - \lambda_2(4z^2+9w^2-36)$$

FOC: $L'_x = z - \lambda_1 \cdot 2x = 0$
$L'_y = w - \lambda_1 \cdot 2y = 0$
$L'_z = x - \lambda_2 \cdot 8z = 0$
$L'_w = y - \lambda_2 \cdot 18w = 0$

C: $x^2+y^2 \le 1$
$4z^2+9w^2 \le 36$

CSC: $\lambda_1 \ge 0$
$\lambda_1(x^2+y^2-1) = 0$
$\lambda_2 \ge 0$
$\lambda_2(4z^2+9w^2-36) = 0$

FOC1: $z = 2\lambda_1 x$

FOC3: $x - 8\lambda_2(2\lambda_1 x) = 0$
$x(1-16\lambda_1\lambda_2) = 0$
$x = 0$ or $\lambda_1\lambda_2 = 1/16$

FOC2: $w = 2\lambda_1 y$

FOC4: $y - \lambda_2 \cdot 18(2\lambda_1 y) = 0$
$y(1-36\lambda_1\lambda_2) = 0$
$y = 0$ or $\lambda_1\lambda_2 = 1/36$

FOC: (a) $x=0$, $y=0$
(b) $x=0$, $\lambda_1\lambda_2 = 1/36$
(c) $\lambda_1\lambda_2 = 1/16$, $y=0$
~~(d) $\lambda_1\lambda_2 = 1/16$, $\lambda_1\lambda_2 = 1/36$~~

(a) $x=0, y=0$: $z=0$, $w=0$ $\Rightarrow \begin{pmatrix} NB \\ NB \end{pmatrix} \overset{CSC}{\Rightarrow} \lambda_1 = \lambda_2 = 0$

$\Rightarrow (0,0,0,0;0,0)$ $f=0$

(b) $x=0$, $\lambda_1\lambda_2 = 1/36$: $z=0$ $\quad \lambda_1,\lambda_2 > 0 \Rightarrow \begin{pmatrix} B \\ B \end{pmatrix}$

$w = 2\lambda_1 y$

$\Rightarrow (0,1,0,2;1,1/36)$ $f=2$ $\Rightarrow \lambda_1 = \dfrac{w}{2y}$

~~$(0,1,0,-2;-1,-1/36)$~~
~~$(0,-1,0,2;-1,-1/36)$~~
$(0,-1,0,-2;1,1/36)$ $f=2$

$x^2+y^2=1$ $\quad y^2=1$
$4z^2+9w^2=36$ $\quad 9w^2=36$
$w^2=4$
$y=\pm 1$
$w=\pm 2$

$\lambda_1 \ge 0 \quad \lambda_2 \ge 0$

c) $\lambda_1\lambda_2 = 1/16,\ y=0$: $\lambda_1, \lambda_2 > 0 \Rightarrow$ (B) (B) $\lambda_1 > 0$, $\lambda_2 \geq$

$w=0$, $z = 2\lambda_1 x$ $\quad x = \pm 1 \quad x^2 = 1 \quad x^2+y^2=1$

$\lambda_1 = \frac{z}{2x} \quad z = \mp 3 \quad 4z^2 = 36 \quad 4z^2+9w^2=36$

$\Rightarrow (1, 0, 3, 0 ; 3/2, 1/24) \quad f = 3$

$\cancel{(1, 0, -3, 0 ; -3/2, -1/24)}$

$\cancel{(-1, 0, +3, 0 ; -3/2, -1/24)}$

$(-1, 0, -3, 0 ; 3/2, 1/24) \quad f = 3$

$f_{max} = 3$?

SOC: $h(x,y,z,w) = L(x,y,z,w ; 3/2, 1/24) = xz + yw - \frac{3}{2}(x^2+y^2-1) - \frac{1}{24}(4z^2+9w^2-36)$

$= \underbrace{-\frac{3}{2}x^2 - \frac{3}{2}y^2 - \frac{1}{6}z^2 - \frac{9}{24}w^2 + xz + yw}_{} + \frac{3}{2} + \frac{3}{2}$ $+3$

$$H(h) = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -3/4 \end{pmatrix}$$

$D_1 = -3$
$D_2 = 9$
$D_3 = -3 \cdot 0 = 0$
$\Delta_2 = 9/4 - 1 = 5/4$

$\Delta_1 = -3, -3, -1/3, -3/4 < 0$

$\Delta_2 = 9, 1, 1/4, 0, 5/4, 9/4 \geq 0$

$\Delta_3 = 0, -5/12, *, * \leq 0$

$\Delta_4 = * = 0 \geq 0$

$$* = \begin{vmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3/4 \end{vmatrix} = -3(9/4 - 1) = -3 \cdot 5/4 = -15/4 < 0$$

$\Downarrow$

h concave

($H(h)$ neg. semidef.)

$\Downarrow$ SOC

$\underline{\underline{f_{max} = 3}}$ at $(1,0,3,0)$, $(-1,0,-3,0)$ $\quad \lambda_1 = 3/2$, $\lambda_2 = 1/24$

KT) std. form

9.1

a) max $f = 2x^2 - 4y^2 - 2z^2$ wh $x^4 + y^4 + z^4 \le 16$

$$L = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - 16)$$

FOC: $L'_x = 4x - \lambda \cdot 4x^3 = 0$ $\quad x^4 + y^4 + z^4 \le 16$ $\quad \lambda \ge 0$

$L'_y = -8y - \lambda\, 4y^3 = 0$ $\quad \lambda(x^4 + y^4 + z^4 - 16) = 0$

$L'_z = -4z - \lambda \cdot 4z^3 = 0$

$4x(1 - \lambda \cdot x^2) = 0$ $\quad \underline{x = 0}$ or $\underline{x^2 = 1/\lambda}$

$4y(-2 - \lambda \cdot y^2) = 0$ $\quad y = 0$ or $\cancel{y^2 = -2/\lambda}$

$4z(-1 - \lambda z^2) = 0$ $\quad z = 0$ or $\cancel{z^2 = -1/\lambda}$

$\Downarrow$

$y = 0, z = 0$ $\to$ $x = 0 \Rightarrow \lambda = 0 \Rightarrow \underline{(0,0,0;0)}$ $f = 0$

$\searrow$ $x \ne 0$, $x^2 = 1/\lambda$ $\quad \lambda > 0$ B $\quad x^4 = 16$

$x^2 = 4$

$x = \pm 2$

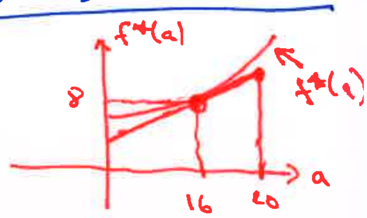
$\Downarrow$

$\underline{(\pm 2, 0, 0; 1/4)}$ $f = 8$

$f_{max} = \underline{\underline{8}}$ at $\underline{(\pm 2, 0, 0)}$ with $\underline{\lambda = 1/4}$

since $\{(x,y,z) : x^4 + y^4 + z^4 \le 16\}$ is compact

+ no adm. pts where NDCQ fails.



b) i) C $\to$ $x^4 + y^4 + z^4 \le 20$: max f wh $x^4 + y^4 + z^4 \le a$

$\underline{a = 16}$: $f^*(16) = \underline{8}$ $\quad x^*(16) = \underline{\pm 2}$ $\quad y^*(16) = 0$, $z^*(16) = \underline{0}$, $\lambda^*(16) = \underline{1/4}$

$\underline{a = 20}$: $f^*(20) \cong \underset{8}{f^*(16)} + \underset{(20-16)}{\Delta a} \cdot \dfrac{df^*(a)}{da}$ Env. Thm. $= L'_a(\underline{x}^*(a); \lambda^*(a))$

$= \lambda^*(a) = \lambda^*(16) = 1/4$

$L = f(x,y,z) - \lambda \cdot (x^4 + y^4 + z^4 - a) = \ldots + \lambda a$

$L'_a = \lambda$

$= 8 + 4 \cdot 1/4 = \underline{\underline{9}}$

ii) $f \to_{b} x^2 - 4y^2 - 2z^2$: max $f = bx^2 - 4y^2 - 2z^2$ when $x^4 + y^4 + z^4 \leq 16$

$\underline{b=2}$: $f^*(2) = \underline{8}$ $x^*(2) = \underline{\pm 2}$ $y^*(2) = z^*(2) = \underline{0}$ $\lambda^*(2) = \underline{1/4}$

$\underline{b=1}$: $f^*(1) \cong \underbrace{f^*(2)}_{8} + \underbrace{\Delta b}_{(1-2)} \cdot \dfrac{df^*(b)}{db}$ — Env. Thm: $\dfrac{df^*(b)}{db} = L'_b(\underline{x}^*(b); \lambda^*(b)) = x^*(b)^2 = 4$

$= 8 - 1 \cdot 4 = \underline{\underline{4}}$

$L = \underline{bx^2} - 4y^2 - \ldots$

$L'_b = x^2$

iii) $\underline{\text{Change } a \to 20 \text{ and } b = 1:}$

$f^*(20,1) = \underbrace{f^*(16,2)}_{8} + \underbrace{\Delta a}_{4} \cdot \underbrace{\dfrac{df^*(a)}{da}}_{1/4} + \underbrace{\Delta b}_{-1} \cdot \underbrace{\dfrac{df^*(b)}{db}}_{(\pm 2)^2}$

$= 8 + 4 \cdot 1/4 - 1 \cdot 4 = \underline{\underline{5}}$

9.3 $\max f(x,y,z,w)$ when $x^2+y^2+z^2+w^2=6$

$$f = \underline{x}^T A \underline{x} \qquad \underline{x}^T \cdot \underline{x} = 6, \quad \underline{x}^T\underline{x} = \underline{x}^T I \underline{x}$$

b) Find solutions of the Lagrange cond. with $\lambda = -12$

$$L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T\underline{x} - 6)$$

FOC: $L'_{\underline{x}} = 2A\underline{x} - \lambda(2I\underline{x}) = \underline{0} \quad :2 \Rightarrow A\underline{x} - \lambda I \underline{x} = \underline{0}$

$\boxed{(A-\lambda I)\underline{x} = \underline{0}}$

C: $\underline{x}^T\underline{x} = 6$

$\|x\|^2 = 6$

$|A-\lambda I| = 0$ or $\cancel{x=0}$

λ is eigenvalue and $\underline{x} \in E_\lambda$.

$\lambda = -12$:

$$A + 12I = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix} + \begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 2 & 2 & 3 & 7 \end{pmatrix} \to \begin{pmatrix} 2 & -2 & 7 & 3 \\ 0 & 2 & -2 & 2 \\ 8 & 0 & 2 & 2 \\ 2 & 2 & 3 & 7 \end{pmatrix} \begin{matrix} -4 \\ \\ \\ -1 \end{matrix}$$

$$\to \begin{pmatrix} 2 & -2 & 7 & 3 \\ 0 & 2 & -2 & 2 \\ 0 & 8 & -26 & -10 \\ 0 & 4 & -4 & 4 \end{pmatrix} \begin{matrix} \\ -4 \\ \\ -2 \end{matrix} \to \begin{pmatrix} 2 & -2 & 7 & 3 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -18 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

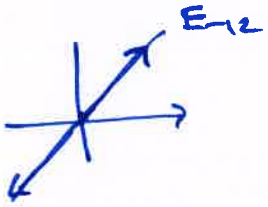
w free, $z = -w$, $2y = 2z - 2w = -4w \Rightarrow y = -2w$

$2x = 2(-2w) - 7(-w) - 3w = 0 \Rightarrow x = 0$

$\Rightarrow (x,y,z,w) = (0,-2w,-w,w) = w \cdot (0,-2,-1,1)$

$\|(0,-2,-1,1)\|^2 = 0^2 + (-2)^2 + (-1)^2 + 1^2 = 6$

$\underline{w = \text{I}\,\text{I}:}$ $(x,y,z,w;\lambda) = (0,-2,-1,1;-12),$
$(0,2,1,-1;-12)$



c) $\underline{\text{SOC}}$: $h = L(x,y,z,w;-12) = \underbrace{\underline{x}^T A \underline{x} + 12(\underline{x}^T\underline{x} - 6)}$

$= \underline{x}^T (A + 12I)\underline{x} - 72$

$H(h) = 2(\underbrace{A + 12I}_{\text{matrix from the previous page}})$

$$\begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 2 & 2 & 3 & 7 \end{pmatrix}$$

$D_1 = 8$
$D_2 = 16$
$D_3 = 8 \cdot 10 + 2 \cdot (-4) = 72$ ✓

$\underline{rk = 3}$ → RCC: pos. semidef.

$f_{min} = f(0,-2,-1,1)$ $\xLeftarrow{\text{SOC}}$ h convex $\Longleftarrow$ pos. semidef.

$= \underline{x}^T A \underline{x}$

$= \underline{x}^T(-12\underline{x})$

$= -12\,\underline{x}^T\underline{x} = -12 \cdot 6 = \underline{\underline{-72}}$

Exam Problems from Plenary Session 3

1) Final exam 11/2019 Q4
2) — " — 06/2018 Q4
3) Key Problem 9.3

① Final exam 11/2019, Q4

$$\min f(x,y,z) = \underbrace{x^2+y^2+z^2-xy+xz-yz}_{\underline{x}^T A \underline{x}} \quad \text{when} \quad x+y+z=11 \qquad \underline{\text{Lagrange}}$$

a)
$$A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix} \qquad H(f) = 2A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

pos. def.

$D_1 = \underline{2}$

$D_2 = \underline{3}$

$D_3 = 1\cdot(-1) - (-1)\cdot(-1) + 2\cdot 3$
$= -1 - 1 + 6 = \underline{4}$

f $\underline{\underline{\text{convex}}}$

b) $L = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 11) \qquad B = (1\ 1\ 1)$

FOC: $L'_{\underline{x}} = \boxed{2A\cdot\underline{x} - \lambda\cdot B^T = \underline{0}}$

C: $\boxed{B\underline{x} = 11}$

$A\underline{x} = \frac{1}{2}\lambda B^T \Rightarrow \underline{x} = A^{-1}(\frac{1}{2}\lambda B^T)$

$\underline{x} = \frac{\lambda}{2}\cdot A^{-1}B^T$

$B\underline{x} = B\cdot\lambda/2\cdot A^{-1}B^T = 11$

$\lambda/2\cdot \underline{BA^{-1}B^T} = 11$

$\Uparrow$

FOC:
$$\begin{aligned} 2x - y + z - \lambda\cdot 1 &= 0 \\ -x + 2y - z - \lambda\cdot 1 &= 0 \\ x - y + 2z - \lambda\cdot 1 &= 0 \end{aligned}$$
C:
$$x + y + z = 11$$

4x4 linear system
⇓
use $\underline{\text{Gauss}}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 2 & -1 & 1 & -1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 1 & -1 & 2 & -1 & 0 \end{array}\right) \begin{matrix} -2 \\ 1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -3 & -1 & -1 & -22 \\ 0 & 3 & 0 & -1 & 11 \\ 0 & -2 & 1 & -1 & -11 \end{array}\right) -1$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 3 & 0 & -1 & 11 \\ 0 & -2 & 1 & -1 & -11 \end{array}\right) \begin{matrix} 3 \\ -2 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & -6 & -1 & -22 \\ 0 & 0 & 5 & -1 & 11 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & -1 & -2 & -11 \\ 0 & 0 & 5 & -1 & 11 \end{array}\right) \big]\cdot 5 \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 11 \\ 0 & -1 & -2 & 0 & -11 \\ 0 & 0 & -1 & -2 & -11 \\ 0 & 0 & 0 & -11 & -44 \end{array}\right)$$

$$\left.\begin{array}{lll} -11\lambda = -44 & & \underline{\lambda = 4} \\ -z-8=-11 & -z=-3 & \underline{z=3} \\ -y-2\cdot 3 = -11 & -y=-5 & \underline{y=5} \\ x+5+3=11 & & \underline{x=3} \end{array}\right\}$$

Cand. pt: $(x,y,z;\lambda) = \underline{(3,5,3;4)}$

$f = 9+25+9-15+9-15 = \underline{22}$

SOC: $h = f(x,y,z) - 4(x+y+z-11)$

$H(h) = H(f)$ pos. defn. from a) $\Rightarrow h$ convex $\xRightarrow{\text{SOC}}$ $f_{min} = \underline{\underline{22}}$ at $(3,5,3)$ with $\lambda = 4$

c) Change C: $x+y+z=10$:

min $f(x,y,z)$ when $x+y+z=a$

$L = f(x,y,z) - \lambda(x+y+z-a)$

$\Rightarrow L'_a = \lambda$ $\xRightarrow{\text{Env. Thm}}$ $\dfrac{df^*(a)}{da} = L'_a(x^*(a); \lambda^*(a)) = \lambda^*(a) = 4$ when $a=11$

$f^*(10) \simeq f^*(11) + \Delta a \cdot \dfrac{df^*(a)}{da}$

$= 22 + (-1)\cdot 4 = \underline{\underline{18}}$

2) Final exam 06/2018, Q4

KT-prob. std. form

max $f(x,y) = xy(x-y)$ when $x^2+y^2+(x-y)^2 \leq 96$

a) $L = xy(x-y) - \lambda(x^2+y^2+(x-y)^2-96)$
$= x^2y - xy^2 - \lambda(x^2+y^2+x^2-2xy+y^2-96)$

$$\left.\boxed{\begin{array}{lll} \text{FOC:} & L'_x = & 2xy - y^2 - \lambda\cdot(2x+2x-2y) = 0 \\ & L'_y = & x^2 - 2xy - \lambda(2y-2x+2y) = 0 \\ \text{C:} & & x^2+y^2+(x-y)^2 \leq 96 \\ \text{CSC:} & & \lambda \geq 0 \text{ and } \lambda\cdot(x^2+y^2+(x-y)^2-96) = 0 \end{array}}\right\} \text{KT-conditions}$$

b) Find all $(x; y; \lambda)$ satisfying FOC + C + CSC with $x, y \neq 0$

FOC: $y(2x-y) - \lambda(4x-2y) = 0$

$x(x-2y) - \lambda(-2x+4y) = 0$

NB

$x^2+y^2+(x-y)^2 < 96$:

CSC: $\lambda = 0$

$y(2x-y) = 0$

$x(x-2y) = 0$

$\Downarrow$

$2x - y = 0$

$x - 2y = 0$

$\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3 \neq 0$

just trivial solution $x = y = 0$

concl: $\boxed{\text{no cand. pts with } x, y \neq 0}$

B

$x^2+y^2+(x-y)^2 = 96$:

CSC: $\lambda \geq 0$

FOC: (1) $\lambda = \dfrac{y(2x-y)}{4x-2y}$ or $2x - y = 0$

$= \dfrac{y\,\cancel{(2x-y)}}{2\,\cancel{(2x-y)}}$

$\lambda = y/2$

(2) $\lambda = \dfrac{x(x-2y)}{-2x+4y}$ or $x - 2y = 0$

$= \dfrac{x\,\cancel{(x-2y)}}{-2\,\cancel{(x-2y)}}$

$\lambda = -\dfrac{x}{2}$

Cases:

(A) $\lambda = y/2 = -x/2$: $y = -x$

$x^2 + (-x)^2 + (2x)^2 = 96$

$6x^2 = 96$

$x^2 = 96/6 = 16$

$x = \pm 4 \quad y = \mp 4 \quad \lambda = \mp 2$

$\Rightarrow$ Cand: $(4, -4; -2)$ $f = -128$

$(-4, 4; 2)$ $f = 128$

(B) $\lambda = y/2$, $x = 2y$:

$(2y)^2 + y^2 + y^2 = 96$

$6y^2 = 96$

$y^2 = 16$

$y = \pm 4 \quad x = \pm 8 \; ; \lambda = \pm 2$

$\Rightarrow$ Cand pts: $(8, 4; 2)$ $f = 128$

$(-8, -4; -2)$ $f = -128$

B (cont'd)

(C) $2x - y = 0$, $\lambda = -x/2$:

$y = 2x \rightarrow x^2 + (2x)^2 + (-x)^2 = 96$

$6x^2 = 96$

$x^2 = 16$

$x = \pm 4$

$\lambda = \mp 2 \quad y = \pm 8$

Cand. pts: $(4, 8; -2)$ $f = -128$

$(-4, -8; 2)$ $f = 128$

(D) $2x - y = 0$, $x - 2y = 0$:

$x = y = 0 \Rightarrow 0 + 0 + 0 \neq 96$

$\Rightarrow$ no points

Concl: six points from

B, case (A), (B), (C)

(c) C: $x^2+y^2+(x-y)^2 \le 96$ compact set since $x^2 \le 96$, $y^2 \le 96$

$\Leftarrow$ EVT $\Rightarrow$

there is a maximum

$-\sqrt{96} \le x \le \sqrt{96}$

$-\sqrt{96} \le y \le \sqrt{96}$

$f_{max} = 128$ since the possible candidate pts are

ordinary cand. pts:

(i) pts $(x;y;\lambda)$ with $x \cdot y \neq 0$ that satisfies FOC+C+CSC
$\Rightarrow$ six points (b), max f-value $= 128$

(ii) pts $(x;y;\lambda)$ with $x=0$ or $y=0$ — " —
$\Rightarrow$ any such point will have $f=0$, **not** max points

iii) Adm points where NDCQ fails: no points

C: $x^2+y^2+(x-y)^2 \le 96$

B: NDCQ: rk $\left(\underbrace{2x+2(x-y)}_{g'_x} \quad \underbrace{2y+2(x-y)(-1)}_{g'_y}\right)=1$

NB: no NDCQ ok

NDCQ fails:

$g'_x = g'_y = 0$

$4x-2y=0$, $-2x+4y=0$

$y=2x$ $\quad -2x+8x=0$

$6x=0$

$y=0$ $\quad x=0$

$\Downarrow$

$(0,0)$ $\quad 0+0+0<96$

not B

③ Key Problem 9.3

max $f(x,y,z,w) = \underline{x}^T A \underline{x}$ when $\underline{x}^T\underline{x} = 6$

d) $L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T\underline{x} - 6)$ $\quad \underline{x}^T \cdot I \underline{x}$

FOC
$$L'(\underline{x}) = 2A\underline{x} - \lambda \cdot (2\underline{x}) = 0 \quad 2I\underline{x}$$
$$2 \cdot [(A - \lambda I)\underline{x}] = 0$$
$$(A - \lambda I)\underline{x} = 0$$

$\cancel{\underline{x} = 0}$ or $|A - \lambda I| = 0$

λ eigenvalue of A

$\underline{x} \in E_\lambda$ eigenvector

C $\qquad \underline{x}^T\underline{x} = 6$

$\Downarrow$

Cond pts: $(\underline{x};\lambda)$ s.t. λ is eigenvalue of A

$\underline{x}$ " eigenvector of A

with $\underline{x}^T\underline{x} = 6$

$$f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T \cdot (\lambda \underline{x})$$
$$= \lambda \cdot \underline{x}^T\underline{x} = \lambda \cdot 6 = \underline{6\lambda}$$

Notice: the highest eigenvalue gives the highest f-val.

From (a): f concave and H(f) negative semidefin.

$\Updownarrow$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq 0$ and at least one $\lambda_i = 0$ $\Big\} \Rightarrow$ the highest eigenvalue $= 0$

$f_{max} = \underline{\underline{0}}$

a) $f(x,y,z,w) = -4x^2 - 10y^2 - 5z^2 - 5w^2$
$+ 4xz + 4xw - 4yz + 4yw + 6zw$

$$A = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix}$$

$D_1 = -4 > 0$

$D_2 = 40 < 0$

$D_3 = -4 \cdot (50 - 4) + 2(0 - (-20))$
$= -4 \cdot 46 + 2 \cdot 20 = -184 + 40 = -144 < 0$

$D_4 = 0$ since $R(4) = -R(3)$

$$\begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix} \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \rightarrow \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 0 & -2 & -4 & 4 \\ 0 & 2 & 4 & -4 \end{pmatrix}$$

$\Rightarrow$ rk $A = 3$, RRC: A neg. semidefinite