	Plan	
1	Key Problems:	10.2b, 10.3b, 10.4b, 11.1b, 11.2cd, 12.2b, 12.4bc, 13.1d, 13.3a
2	Final Exams:	01/2020 2b,3b 11/2019 3b 01/2019 2b

(1) Key Problems

- linear
- separable
- exact

10.2b. $y' = y^2 \cdot t$

$$\frac{1}{y^2} y' = t$$

$$y^{-2} \quad \int \frac{1}{y^2}\,dy = \int t\,dt$$

$$\frac{y^{-1}}{-1} = \frac{1}{2}t^2 + C$$

$$\frac{1}{y} = -\frac{1}{2}t^2 - C \qquad y = \frac{1 \cdot (-2)}{\left(-\frac{1}{2}t^2 - C\right) \cdot (-2)}$$

$$y = \underline{\underline{\frac{-2}{t^2 + 2C}}}$$

10.3c. $y' \underbrace{-2t}_{a(t)} y = \underbrace{4t}_{b(t)}$

Int. factor: $u = e^{\int a(t)dt} = e^{\int -2t\,dt} = \underline{e^{-t^2}}$

$$(y \cdot e^{-t^2})' = 4t \cdot e^{-t^2}$$

$$y \cdot e^{-t^2} = \int 4t e^{-t^2} = \int 4t \cdot e^u \cdot \frac{du}{-2t} \qquad \boxed{\begin{array}{l} u = -t^2 \\ du = -2t\,dt \end{array}}$$

$$= \int -2e^u\,du = -2e^u + C$$

$$y \cdot e^{-t^2} = -2e^{-t^2} + C \Rightarrow y = \underline{\underline{-2 + Ce^{t^2}}}$$

10.4 b $\underbrace{2y-3t^2}_{h'_t} + \underbrace{2(y+t)}_{h'_y}\, y' = 0$

① $h'_t = 2y - 3t^2$

② $h'_y = 2y + 2t$

① $h = 2yt - t^3 + \alpha(y)$

② $(2yt - t^3 + \alpha(y))'_y = 2y + 2t$

$\cancel{2t} - \cancel{0} + \alpha'(y) = 2y + \cancel{2t}$

$\alpha(y) = y^2$

the equation is exact, general solution is

$$h(t,y) = C$$

$$2ty - t^3 + y^2 = C$$

$$y^2 + (2t)\cdot y + (-t^3 - C) = 0$$

$$y = \frac{-2t \pm \sqrt{(2t)^2 - 4\cdot 1\,(-t^3-C)}}{2\cdot 1}$$

$$= -t \pm \frac{1}{2}\sqrt{4t^2 + 4t^3 + 4C}$$

$$= -t \pm \sqrt{t^2 + t^3 + C}$$

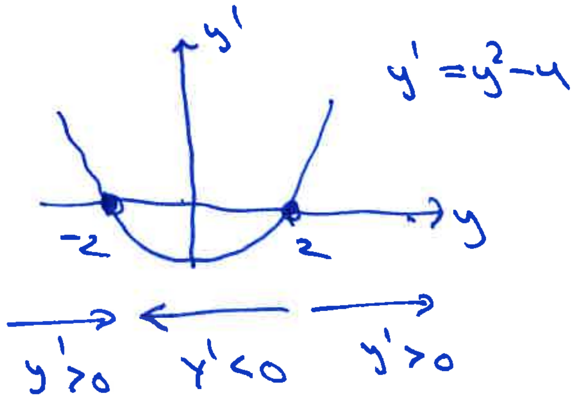
11.1 b $\quad y' = y^2 - 4$

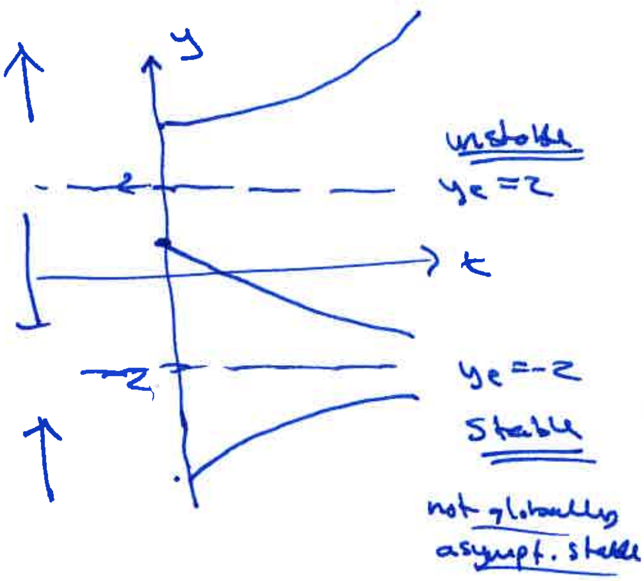
Eq. states: $y^2 - 4 = 0$

$y^2 = 4$

$y = \pm 2 \quad \Rightarrow y_e = 2$ and $y_e = -2$

Stability:





11.2 e $\quad y'' - 3y' + 2y = 3e^{2t}$

Superposition: $y = y_h + y_p = C_1 e^{2t} + C_2 e^{t} + \cancel{\tfrac{3}{2}e^{3t}}\ 3te^{2t}$

y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 2,\ r = 1 \quad \Rightarrow y_h = c_1 e^{2t} + c_2 \cdot e^{t}$

y_p: $y'' - 3y' + 2y = 3e^{2t}$

$$\begin{cases} y = Ae^{2t} \\ y' = Ae^{3t}\cdot 2 \\ y'' = Ae^{3t}\cdot 4 \end{cases}$$

$4Ae^{2t} - 3\cdot(2Ae^{2t}) + 2(Ae^{2t}) = 3e^{3t}$

$\cancel{(9A - 9A + 2A)e^{3t} = 3e^{3t}}$

$\cancel{2A = 3}$

$\cancel{A = 3/2 \Rightarrow y_p = \tfrac{3}{2}e^{3t}}$

$(4A - 6A + 2A)e^{2t} = 3e^{2t}$

$0A \cdot e^{2t} = 3e^{2t}$

not possible

$f = 3e^{2t}$

$f' = 6e^{2t}$

$f'' = 12e^{2t}$

$\Rightarrow y = Ae^{2t}$

Next: Try $y = Ae^{2t} \cdot t = \underline{Ate^{2t}}$

$$y' = A \cdot e^{2t} + At \cdot 2e^{2t} = \underline{(2At + A)e^{2t}}$$

$$y'' = 2A \cdot e^{2t} + (2At + A) \cdot 2e^{2t}$$

$$= \underline{(4A + 4At)e^{2t}}$$

$$y'' - 3y' + 2y = 3e^{2t}$$

$$(4A + 4At)e^{2t} - 3(2At + A)e^{2t} + 2(At)e^{2t} = 3e^{2t}$$

$$(4A - 3A)e^{2t} + \cancel{(4A - 6A + 2A)te^{2t}} = 3e^{2t}$$

$$Ae^{2t} = 3e^{2t} \qquad \underline{\underline{A = 3}}$$

$$y_p = \underline{3te^{2t}}$$

10.2d) $y'' - y = t^2$

Superpos: $y = y_h + y_p = \underline{\underline{C_1 e^t + C_2 e^{-t} - t^2 - 2}}$

y_h: $r^2 - 1 = 0$
$r = \pm 1 \quad \Rightarrow y_h = C_1 e^t + C_2 e^{-t}$

y_p:
$$\left.\begin{aligned} y &= At^2 + Bt + C \\ y' &= 2At + B \\ y'' &= 2A \end{aligned}\right\} \quad 2A - (At^2 + Bt + C) = t^2$$

$$\underset{1}{\underset{\shortparallel}{(-A)}}t^2 + \underset{0}{\underset{\shortparallel}{(-B)}}t + \underset{0}{\underset{\shortparallel}{(2A - C)}} = t^2$$

$$A = -1 \qquad B = 0 \qquad 2 \cdot (-1) - C = 0$$
$$C = -2$$

$$y_p = -1 \cdot t^2 - 2 = \underline{-t^2 - 2}$$

12.2 b)

$$y' = \underbrace{\begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix}}_{\text{"}A\text{"}} \cdot y \, , \quad y(0) = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

$$y = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + c_3 \underline{v_3} e^{\lambda_3 t}$$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & 0 \\ 3 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \cdot \left[(2-\lambda)(1-\lambda) - 3 \right] + 1 \cdot \left[1-\lambda+1 \right] = 0$$

$$(2-\lambda) \cdot (\lambda^2 - 3\lambda - 1) + (2-\lambda) = 0$$

$$(2-\lambda) \cdot (\lambda^2 - 3\lambda \cancel{-1} \cancel{+1}) = 0$$

$$(2-\lambda) \cdot \lambda \cdot (\lambda - 3) = 0$$

$$\underline{\lambda = 2} \quad \underline{\lambda = 0} \quad \underline{\lambda = 3}$$

E_2:

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 3 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{v_1} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

E_0:

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & +1 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \Big] 2 \to \begin{pmatrix} -1 & 2 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{v_2} = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$$

E_3:

$$-1 \Big[\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 3 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{v_3} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

General solution:

$$y = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \cdot \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} e^{3t}$$

t=0:

$$c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + c_3 \cdot \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & -2 & -1 & -1 \\ -1 & -1 & 1 & -3 \\ 1 & 5 & -2 & 8 \end{array} \right) \to \left(\begin{array}{ccc|c} -1 & -1 & 1 & -3 \\ 0 & -2 & -1 & -1 \\ 1 & 5 & -2 & 8 \end{array} \right) \Big] 1 \to \left(\begin{array}{ccc|c} -1 & -1 & 1 & -3 \\ 0 & -2 & -1 & -1 \\ 0 & 4 & -1 & 5 \end{array} \right) \Big] 2$$

$$\to \left(\begin{array}{ccc|c} -1 & -1 & 1 & -3 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -3 & 3 \end{array} \right)$$

$$c_1 = 1 \qquad c_2 = 1 \qquad c_3 = -1$$

Particular solution:

$$y = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{3t}$$

12.4 bc):

$$y(t) = \underbrace{3e^{-2t} - 5e^{t} + 12e^{-3t}}_{y_h} \qquad y_p = 0$$

b) $y''' + ay'' + by' + cy = f(t) = 0$

$r = -2, 1, -3$ are roots in the char. eqn.

$$(r+2)(r-1)(r+3) = (r^2 + r - 2)(r+3)$$
$$= r^3 + 4r^2 + r - 6$$

$\Downarrow$

$$y''' + 4y'' + y' - 6y = 0$$

c)

$y_1 = y \qquad y_1' = y_2$

$y_2 = y' \qquad y_2' = y_3$

$y_3 = y'' \qquad y_3' = y''' = 6y - y' - 4y''$

$$= 6y_1 - y_2 - 4y_3$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

13.1d) $y_{t+2} + y_{t+1} - 2y_t = 6$

Superposition: $y_t = y_t^h + y_t^p = \underline{C_1(-2)^t + C_2 + 2t}$

y_t^h: $y_{t+2} + y_{t+1} - 2y_t = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r = -2,\ r = +1 \implies y_t^h = C_1 \cdot (-2)^t + C_2 \cdot (1)^t$

$= C_1 \cdot (-2)^t + C_2$

y_t^p: $\left.\begin{array}{l} y_t = A \\ y_{t+1} = A \\ y_{t+2} = A \end{array}\right\}$ $A + A - 2A = 6$

$0 \cdot A = 6$

impossible.

Next:

$\left.\begin{array}{l} y_t = At \\ y_{t+1} = A(t+1) = At + A \\ y_{t+2} = A(t+2) = At + 2A \end{array}\right\}$ $(At + 2A) + (At + A) - 2(At) = 6$

$(\cancel{At + At - 2At}) + 3A = 6$

$0\,At$

$A = 2$

$y_t^p = 2t$

133 a) $p_{t+2} - 2p_{t+1} + p_t = -15$, $p_0 = 695$, $p_1 = 743$

Superpos: $p_t = p_t^h + p_t^p = c_1 + c_2 t - 7.5t^2$

p_t^h: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r_1 = r_2 = 1 \longrightarrow p_t^h = c_1 \cdot 1^t + c_2 \cdot t \cdot 1^t$

$= c_1 + c_2 t$

p_t^p:

$p_t = A$, $p_{t+1} = A$, $p_{t+2} = A$ $\quad$ $A - 2A + A = -15$

$0A = -15$

impossible

$p_t = At$, $p_{t+1} = At + A$, $p_{t+2} = At + 2A$ $\quad$ $(At + 2A) - 2(At + A) + At = -15$

$(A - 2A + A)t + (2A - 2A) = -15$

$0 = -15$

impossible

$p_t = At^2$

$p_{t+1} = A(t+1)^2 = At^2 + 2At + A$

$p_{t+2} = A \cdot (t+2)^2 = At^2 + 4At + 4A$

$(At^2 + 4At + 4A) - 2(At^2 + 2At + A) + At^2 = -15$

$0 \cdot At^2 + 0At + (4A - 2A) = -15$

$2A = -15$

$A = -\frac{15}{2}$

$y_t^p = -7.5t^2$

$p_0 = 695$: $c_1 = 695 \leftarrow t = 0$

$p_1 = 743$: $c_1 + c_2 - 7.5 = 743 \leftarrow t = 1$

$c_1 = 695$

$c_2 = 55.5$

$\Downarrow$

$y_t = 695 + 55.5t - 7.5t^2$

(2) Final exams:

01/2020, 2b:

$$A = \begin{pmatrix} -7 & 6 & 2 \\ -6 & 5 & 2 \\ -6 & 6 & 1 \end{pmatrix} \quad \underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2b) $A\underline{v_1} = \lambda_1 \underline{v_1}$:

$$A \cdot \underline{v_1} = \begin{pmatrix} -7 & 6 & 2 \\ -6 & 5 & 2 \\ -6 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7\cdot 1 + 6 \cdot 1 + 2 \cdot 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} = \underset{1}{\underset{\parallel}{\lambda_1}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Downarrow$

$\underline{v_1}$ is an ~~eigenvekto~~ eigenvector of A with eigenvalue $\underline{\underline{\lambda = 1}}$

3b) $e^t y' = t y^2$

- ~~linear~~
- separable
- exact

$$y' = \frac{ty^2}{e^t} = \underline{ty^2 \cdot e^{-t}}$$

$$y' = y^2 \cdot te^{-t}$$

separable

$$\int u'v\,dt = uv - \int uv'\,dt$$

$u = -e^{-t} \quad v = t$

$u' = e^{-t} \quad v' = 1$

$$\frac{1}{y^2} y' = te^{-t}$$

$y^{-2} \longrightarrow$

$$\int \frac{1}{y^2}\,dy = \int te^{-t}\,dt$$

$$\frac{y^{-1}}{-1} = -e^{-t} \cdot t - \int -e^{-t} \cdot 1\,dt$$

$$-\frac{1}{y} = -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C$$

$$\frac{1}{y} = te^{-t} + e^{-t} - C \Rightarrow y = \frac{1}{te^{-t} + e^{-t} - C} \frac{\cdot e^t}{\cdot e^t} = \underline{\underline{\frac{e^t}{t + 1 - Ce^t}}}$$

4. $\min f(x,y,z,w) = -4x^2 - 10y^2 - 5z^2 - 5w^2 + 4xz + 4xw - 4yz + 4yw + 6zw$

subj. to $g(x,y,z,w) = x^2 + y^2 + z^2 + w^2 = 6$

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} = I$$

a) $f(\underline{x}) = \underline{x}^T A \underline{x}$

$f'(\underline{x}) = 2A \cdot \underline{x}$

$H(x) = 2A$

$|H(x)| = |2A| = 0$

$= 2^4 \cdot |A| = 0$

$\Downarrow$

$|A| = 0$

$$A = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix}$$

$D_1 = \underline{-4}$

$D_2 = \underline{40}$

$D_3 = -4(50-4) + 2(0+20)$

$= -4 \cdot 46 + 40 = -184 + 40$

$= \underline{-144}$

$D_4 = |A| = 0$

$\Rightarrow$ RRC $\text{rk} A = 3$ $\quad A$ neg. semidefn.

$\Downarrow$

f concave

b) Find all $(x,y,z,w;-12)$ that solves FOC+C

$L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T \cdot I \cdot \underline{x} - 6) = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T \underline{x} - 6)$

FOC: $L'(\underline{x}) = 2A\underline{x} - \lambda(2I\underline{x}) = 2A\underline{x} - \lambda(2\underline{x}) = 0$

C: $\underline{x}^T \underline{x} = 6$

$\lambda = -12$: $2A\underline{x} + 24\underline{x} = 0 \Rightarrow 2A\underline{x} = -24\underline{x}$

$A\underline{x} = -12\underline{x}$

$\Downarrow$

$\underline{x} = 0$ or $\underline{x}$ is in E_{-12}

$\underline{x}^T \cdot \underline{x} = 0 \neq 6$

impossible

$$E_{-12}: \begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 2 & 2 & 3 & 7 \end{pmatrix}$$

$A - \lambda I$ with $\lambda = -12$

$$\begin{pmatrix} 2 & 2 & 3 & 7 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 8 & 0 & 2 & 2 \end{pmatrix} \begin{matrix} \\ \\ -1 \\ -4 \end{matrix} \to \begin{pmatrix} 2 & 2 & 3 & 7 \\ 0 & 2 & -2 & 2 \\ 0 & -4 & 4 & -4 \\ 0 & -8 & -10 & -26 \end{pmatrix} \begin{matrix} \\ \\ 2 \\ 4 \end{matrix}$$

$$\to \begin{pmatrix} 2 & 2 & 3 & 7 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -18 & -18 \end{pmatrix}$$

w free

$z = -w$

$2y - 2(-w) + 2w = 0$

$2y = -4w \Rightarrow y = -2w$

$2x + 2(-2w) + 3(-w) + 7w = 0$

$2x = 4w + 3w - 7w = 0$

$\Rightarrow x = 0$

E_{-12}: $(x, y, z, w) = (0, -2w, -w, w)$

$= w(0, -2, -1, 1)$

C: $\mathbf{x}^T\mathbf{x} = (0 \ -2w \ -w \ w)\begin{pmatrix} 0 \\ -2w \\ -w \\ w \end{pmatrix} = 4w^2 + w^2 + w^2 = 6w^2$

$= 6$

$\Downarrow$

$w^2 = 1$

$w = \pm 1$

$\mathbf{x}^T\mathbf{x} = w \cdot (0 \ -2 \ -1 \ 1) \cdot w\begin{pmatrix} 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}$

$= w^2 \cdot 6$

candidate pts with $\lambda = -12$:

$w = 1$: $(0, -2, -1, 1; -12)$

$w = -1$: $(0, 2, 1, -1; -12)$

c) Use SOC: $h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; -12)$

$= \mathbf{x}^T A \mathbf{x} + 12(\mathbf{x}^T\mathbf{x} - 6)$

$= \mathbf{x}^T A \mathbf{x} + 12(\mathbf{x}^T I \mathbf{x}) - 72$

$= \mathbf{x}^T A \mathbf{x} + \mathbf{x}^T(12I)\mathbf{x} - 72$

$= \mathbf{x}^T(A + 12I)\mathbf{x} - 72$

$H(h) = 2(A + 12I)$ ✓

Must show that $A + 12I$ is pos. semidef.

$$A + 12I = \begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 2 & 2 & 3 & 7 \end{pmatrix}$$

pos. semidef ?

$D_1 = 8$

$D_2 = 16$

$D_3 = 8 \cdot 10 + 2 \cdot (-4) = 72$

$D_4 = 0$

$\Downarrow$ RRC $(\text{rk } A = 3)$

$A + 12I$ pos. semidef.

$\Downarrow$

h convex

$\Downarrow$

The two cand. pts are <u>min pts</u>