Plan
1 Introduction to first order differential equations
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4 Exact differential equations

Plenary Session 3

Mon 31/10

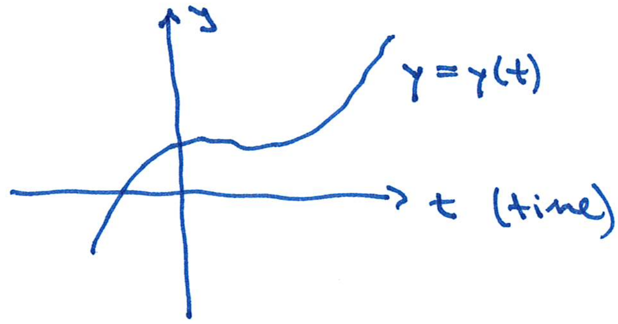
Lecture 7–9

① Introduction to differential equations

Ex: $y' = 2t-2$ first order   unknown function $y = y(t)$

$ty' = y-1$ first order

$y'' = y \cdot y'$ second order diff. eqn.



Ex: Simple integration

$y' = 2t-2$

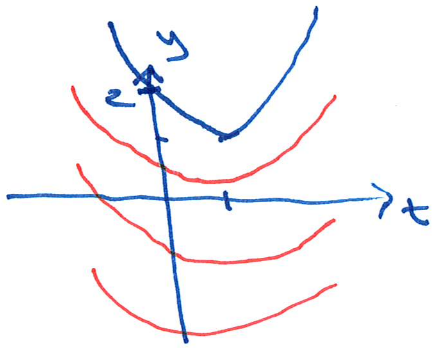
$y = \int 2t-2 \, dt = t^2 - 2t + C$

General solution: $y = \underline{\underline{t^2-2t+C}}$ undetermined coefficient

Initial condition: $y(0) = 2$

$\Updownarrow$

$t = 0, y = 2$



Particular solution:

$y = t^2-2t+C$

$2 = 0^2 - 2 \cdot 0 + C$ ⟹ $C = 2$

$y = \underline{\underline{t^2-2t+2}}$

$= (t-1)^2 + 1$

Defn: A first order differential equation is an eqn. that involves y', y, t (first order, but not higher order derivatives)

Std. form: $y' = F(t,y)$

Fact: Any first order differential equation has general solution that depends on one undetermined coeff. and we need one initial condition to find a unique solution (particular solution).

② Separable differential equations

Defn: A first order diff. eqn. is separable if it can be written in the form

$$y' = f(t) \cdot g(y)$$

Ex: $y' = y + t$ not sep.

$$ty' = 2y - 4 \implies y' = \frac{2y-4}{t} = \frac{2(y-2)}{t} = \underbrace{\frac{2}{t}}_{f(t)} \cdot \underbrace{(y-2)}_{g(y)}$$

Sep.

$$y' = y \cdot (1 - y/10) = \underbrace{1}_{f(t)} \cdot \underbrace{y(1-y/10)}_{g(y)}$$

Solution method:

$$y' = f(t) \cdot g(y) \quad | : g(y)$$

$$\frac{1}{g(y)} y' = f(t) \quad | \int \_\_ \, dt$$

$$\int \frac{1}{g(y)} y' \, dt = \int f(t) \, dt$$

"Substitution"
$dy = y' dt$

$$\boxed{\int \frac{1}{g(y)} dy = \int f(t) \, dt}$$

Ex: $ty' = 2y - 4$

$$y' = \frac{2}{t} \cdot (y-2)$$

$$\frac{1}{y-2} y' = \frac{2}{t}$$

$$\int \frac{1}{y-2} dy = \int \frac{2}{t} dt$$

$$\ln|y-2| + C_1 = 2 \ln|t| + C_2$$

$$\ln|y-2| = 2 \ln|t| + C$$

implicit solution (general)

$$\ln|y-2| = 2\ln|t| + C$$ Solve for y

$$e^{\ln|y-2|} = e^{2\ln|t|+C}$$

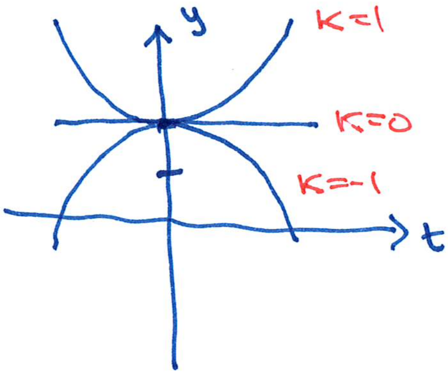
$$|y-2| = e^{2\ln|t|} \cdot e^{C} = e^{\ln|t|^2} \cdot e^{C}$$

$$|y-2| = e^{C} \cdot |t|^2$$

$$|y-2| = e^{C} \cdot t^2$$

$$y-2 = \pm e^{C} \cdot t^2 = Kt^2$$

$$y = 2 + Kt^2$$ general solution in explicit form



Ex: $y' = 2y$

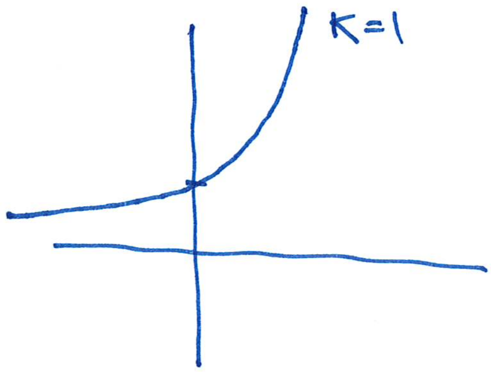
$$\frac{1}{y} y' = 2$$

$$\int \frac{1}{y}\,dy = \int 2\,dt$$

$$\ln|y| = 2t + C$$

$$|y| = e^{2t+C} = e^{2t} \cdot e^{C}$$

$$y = \pm e^{C} \, e^{2t} = Ke^{2t}$$



General solution: $y = Ke^{2t}$

③ Linear first order differential equations

Defn: A first order diff. eqn. is linear if it can be written in the form

$$\boxed{y' + a(t)\cdot y = b(t)} \iff y' = \underbrace{b(t) - a(t)\cdot y}_{\text{linear in } y}$$

std. form

Ex: $y' = y + t \Rightarrow y' - y = t$ linear with $\begin{cases} a(t) = -1 \\ b(t) = t \end{cases}$

$ty' = 2y - 4 \Rightarrow y' = \frac{2y-4}{t} = \frac{2}{t}\cdot(y-2)$ separable

$y' = \frac{2}{t}\cdot y - \frac{4}{t}$ linear

$y' - \frac{2}{t}y = -\frac{4}{t}$ $\quad a(t) = -2/t$, $b(t) = -4/t$

$t^2 + y^2 \cdot y' = 0$ not linear $\quad y' = -\frac{t^2}{y^2} = -t^2\cdot\frac{1}{y^2}$

separable

Solution method: Integrating factor

$u(t)$ = integrating factor must satisfy:

$y' + a(t)y = b(t) \quad | \cdot u(t) = u$

$\underbrace{uy' + a(t)\cdot u\cdot y}$ $= b(t)\cdot u$

$u'y = a(t)\cdot u\cdot y$

$\Updownarrow$

$\underline{u' = a(t)\cdot u}$ why? separable

$uy' + u'y$

$= (u\cdot y)' = b(t)\cdot u \quad | \int \ldots dt$

$u\cdot y = \int b(t)\cdot u(t)\, dt$

$$\boxed{u = e^{\int a(t)dt}}$$

$$\boxed{y = \frac{1}{u(t)}\cdot\int b(t)\cdot u(t)\, dt}$$

$\frac{1}{u}\cdot u' = a(t)$

$\int \frac{1}{u}\, du = \int a(t)\, dt$

$\ln|u| = \int a(t)\, dt$

Ex: $y' = y + t$

$y' - y = t \quad | \cdot e^{-t}$ $\quad a(t) = -1$, $b(t) = t$

$e^{-t}y' - e^{-t}y = te^{-t}$

$(e^{-t} \cdot y)' = te^{-t}$

$\boxed{\text{Int. factor: } \int a(t)\,dt = \int -1\,dt = -t + C \qquad u = e^{-t} \quad (C=0)}$

$e^{-t} \cdot y = \int te^{-t}\,dt = t \cdot (-e^{-t}) - \int 1 \cdot (-e^{-t})\,dt$

Int. by parts $\quad \int uv'\,dt = uv - \int u'v\,dt$

$\boxed{\begin{array}{ll} u = t & v = -e^{-t} \\ u' = 1 & v' = e^{-t} \end{array}}$

$e^{-t}y = -te^{-t} + \int e^{-t}\,dt = \underline{-te^{-t} - e^{-t} + C}$

$e^{t} \cdot | \quad \rightarrow \quad y = \underline{\underline{-t - 1 + Ce^{t}}}$ general solution

Ex: $ty' = 2y - 4 \qquad y' = \frac{2}{t}y - \frac{4}{t} \qquad \boxed{y' - \frac{2}{t}y = -\frac{4}{t}}$

$a(t) = -2/t \quad b(t) = -4/t$

Int. factor:

$\int -2/t\,dt = -2\ln|t| + C$

$u = e^{-2\ln|t|} = e^{\ln|t|^{-2}}$

$= |t|^{-2} = \frac{1}{|t|^2} = \underline{\frac{1}{t^2}}$

$\left(\frac{1}{t^2} \cdot y\right)' = -\frac{4}{t} \cdot \frac{1}{t^2}$

$\frac{1}{t^2}y = \int -\frac{4}{t^3}\,dt = -4\int t^{-3}\,dt$

$t^2 \cdot | \quad \frac{1}{\cancel{t^2}}y = -4\,\frac{t^{-2}}{(-2)} + C = 2 \cdot \frac{1}{t^2} + C$

$y = \underline{\underline{2 + Ct^2}}$

Power rule: $(n \neq -1)$

$\int t^n\,dt = \frac{t^{n+1}}{n+1} + C$

3 Exact differential equations

Defn: A first order diff. eqn. is **exact** if it can be written in the form

$$p(t,y) + q(t,y) \cdot y' = 0 \qquad (p + q \cdot y' = 0)$$

such that $p(t,y) = h'_t$ and $q(t,y) = h'_y$ for some function $h = h(t,y)$.

Solution method: $\boxed{h(t,y) = C}$

Ex: $t^2 + y^2 \cdot y' = 0$

$$\begin{aligned} p &= \boxed{t^2 = h'_t} \quad (1) \\ q &= \boxed{y^2 = h'_y} \quad (2) \end{aligned}$$

(1) $h'_t = t^2 \Rightarrow h = \underline{\tfrac{1}{3}t^3 + Q(y)}$ $\leftarrow$ $h = \int p(t,y)\,dt = \int t^2\,dt = \tfrac{1}{3}t^3 + Q(y)$

(2) $\underline{h = \tfrac{1}{3}t^3 + Q(y)}$: $y^2 = 0 + Q'(y)$

$Q(y) = \int y^2\,dy \rightarrow y^2 = Q'(y)$

$Q(y) = \tfrac{1}{3}y^3 + C$

$\Rightarrow h = \tfrac{1}{3}t^3 + \tfrac{1}{3}y^3 + \cancel{C}$

Concl: – the diff. eqn is exact.

– general solution:

$h(t,y) = C$

$\tfrac{1}{3}t^3 + \tfrac{1}{3}y^3 + C_1 = C_2$

$\underline{\tfrac{1}{3}t^3 + \tfrac{1}{3}y^3 = C}$

$\tfrac{1}{3}y^3 = C - \tfrac{1}{3}t^3 \quad | \cdot 3$

$y^3 = 3C - t^3$

$y = \sqrt[3]{3C - t^3} = \underline{\underline{\sqrt[3]{K - t^3}}}$

Why? $p + q \cdot y' = 0$ exact $p = h'_t, \; q = h'_y$

$$\underbrace{h'_t + h'_y \cdot y'}_{\frac{dh}{dt}} = 0 \Leftrightarrow \frac{dh}{dt} = 0 \Rightarrow \underline{h(t,y) = C}$$

$\frac{dh}{dt}$ = the total derivative

$$h = \frac{1}{3}t^3 + \frac{1}{3}y^3 : \quad \frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = t^2 + y^2 \cdot y'$$

with $y = y(t)$ total derivative

Ex: $y' = \frac{3t^2 - y^2}{2ty} \quad | \cdot 2ty$

Exact? $p + qy' = 0$

yes

$$2ty \cdot y' = 3t^2 - y^2$$

$$\underbrace{y^2 - 3t^2}_{p} + \underbrace{2ty}_{q} \cdot y' = 0$$

requirements: (for exactness)

$$\boxed{\begin{aligned} &(1) \quad y^2 - 3t^2 = h'_t \\ &(2) \quad 2ty = h'_y \end{aligned}}$$

(1) $h'_t = y^2 - 3t^2$

$h = \int y^2 - 3t^2 \, dt$

$h = \underline{y^2 t - t^3 + C(y)}$

(2) $2ty = \left(y^2 t - t^3 + C(y)\right)'_y$

$2ty = 2yt - 0 + C'(y)$

$2ty = 2ty + C'(y)$

$0 = C'(y)$

$C(y) = 0$

$h = y^2 t - t^3 + 0$

$= \underline{y^2 t - t^3}$

General solution:

$h = y^2 t - t^3 = C$

$$\frac{y^2 t}{t} = \frac{C + t^3}{t}$$

$$y^2 = \frac{C + t^3}{t}$$

$$y = \pm \sqrt{\frac{C + t^3}{t}}$$