Plan

1 Equilibrium states and stability

2 Linear second order differential equations

Ex: $y' = 5y(1 - y/10)$ not linear

Review:
- separable
- linear
- exact

$$= \underbrace{5}_{f(t)} \underbrace{y \cdot \frac{10-y}{10}}_{g(y)}$$ separable

$$\frac{10}{y(10-y)} \cdot y' = 5 \quad \Big| \int \text{-} \, dt$$

$$\int \frac{10}{y(10-y)} \, dy = \int 5 \, dt = 5t + C$$

$$\frac{10}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y}$$

$$10 = A(10-y) + By$$

$$10 = \underset{\overset{\|}{10}}{10A} + \underset{\overset{\|}{0}}{(-A+B)}y$$

$$10A = 10 \quad \underline{A = 1}$$

$$-A + B = 0 \quad B = A = \underline{1}$$

$$\int \frac{1}{y} + \frac{1}{10-y} \, dy = \underline{\ln|y| - \ln|10-y|} = 5t + C$$

$$\ln\left|\frac{y}{10-y}\right| = 5t + C$$

$$\left|\frac{y}{10-y}\right| = e^{5t+C} = e^C \cdot e^{5t}$$

$$(10-y) \cdot \Big| \quad \frac{y}{10-y} = K \cdot e^{5t} \quad (K = \pm e^C)$$

$$y = Ke^{5t} \cdot (10-y) = 10Ke^{5t} - Ke^{5t}y$$

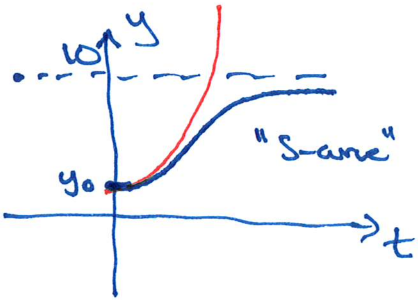
$$y(1 + Ke^{5t}) = 10 \cdot Ke^{5t}$$

$$\underline{\underline{y = 10 \cdot \frac{Ke^{5t}}{1 + Ke^{5t}}}}$$

$$y_0 = 10 \cdot \frac{K}{1+K}$$

$y' = \underline{5}y(1 - y/\underline{10})$

logistic diff. eqn.



$y' = 5y$

$\underline{y = y_0 \cdot e^{5t}}$

① Equilibrium states and stability

Defn: A first order diff-eqn. is called <u>autonomous</u> if it can be written $\boxed{y' = F(y)}$

Defn: An equilibrium state <u>ye</u> is a value of y such that $F(y_e) = 0$.

Ex: $y' = 5y(1 - y/10)$ **Eq. states:** $5y(1-y/10) = 0$

$y = 0$ or $y = 10$

$\Downarrow$

$\underline{y_e = 0}$, $\underline{y_e = 10}$

Idea: If $y_0 = y_e$ is an eq. state, then $y(t) = y_e$ is a constant solution

Eq. states = constant solutions

Defn: An equilibrium state ye is <u>stable</u> if

y_0 close to $y_e \Rightarrow y(t) \to y_e$ as time passes

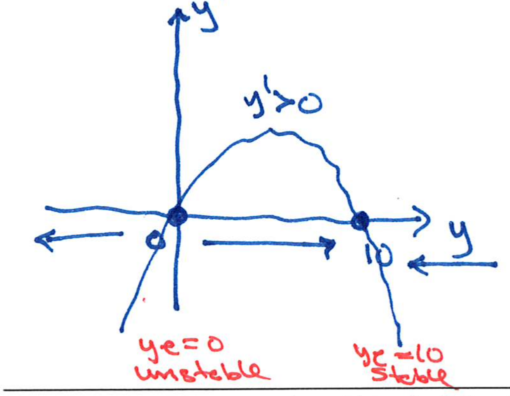
and it is <u>unstable</u> if

y_0 close to $y_e \Rightarrow y(t)$ moves away from the eq. state

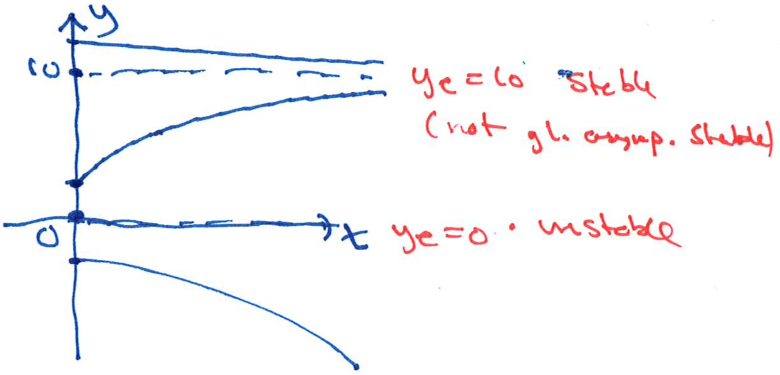
Ex: $y' = 5y(1 - y/10)$ **Eq. states:** $\underline{y_e = 0}$, $\underline{y_e = 10}$

$= 5y - \frac{1}{2}y^2$

Phase diagram:



Solution curves:



Stability Thm:

$y = y_e$ eq. state of $y' = F(y)$: $\boxed{\begin{array}{l} F'(y_e) > 0 \Rightarrow y_e \text{ unstable} \\ F'(y_e) < 0 \Rightarrow y_e \text{ stable} \end{array}}$

Ex: $y' = 5y(1 - y/10)$

$= \underbrace{5y - \frac{1}{2}y^2}_{F(y)}$

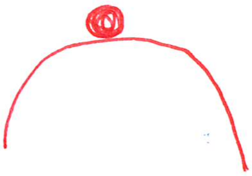
$F'(y) = 5 - y$

$y_e = 0$: $F'(0) = 5 > 0$ unstable

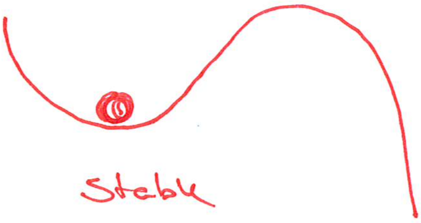
$y_e = 10$: $F'(10) = -5 < 0$ stable

Def: An eq. state is globally asymptotically stable if $y(t) \to y_e$ for any y_0.

globally asymptotically stable



unstable



stable

i) The homogeneous case: $y''+ay'+by=0$

In general:

$y''+ay'+by=0$

$\Updownarrow$

Char. eqn:

$r^2+ar+b=0$

$r=\frac{-a\pm\sqrt{a^2-4b}}{2}$ char. roots

$\Delta = a^2-4b$: discriminant

a) $\Delta>0$: two distinct roots $r_1 \neq r_2$

$\Rightarrow y = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$ general solutions

b) $\Delta=0$: $r_1=r_2=-a/2$ double root

$\Rightarrow y = C_1 e^{-a/2 \cdot t} + C_2 t e^{-a/2 \cdot t}$

$= (C_1+C_2 t)e^{-\frac{a}{2}t}$ general solution

c) $\Delta<0$: no real roots

$r=\frac{-a\pm\sqrt{a^2-4b}}{2} = \underbrace{-\frac{a}{2}}_{\alpha} \pm \underbrace{\frac{\sqrt{4b-a^2}}{2}}_{\beta}\cdot\sqrt{-1}$

$\Rightarrow y = e^{\alpha t}\cdot\left(C_1\cos(\beta t)+C_2\sin(\beta t)\right)$ general solution

Ex: $y''-3y'+2y=0$

$r^2-3r+2=0$

$(r-1)(r-2)=0$

$r_1=1$, $r_2=2$

$r=\frac{3\pm\sqrt{9-4\cdot 2}}{2}=\frac{3\pm 1}{2}$

General solution:

$y=C_1e^{t}+C_2e^{2t}$

Ex: $y''-4y'+4y=0$

$r^2-4r+4=0$

$(r-2)^2=0$ $r_1=r_2=2$

General solution:

$y=C_1e^{2t}+C_2te^{2t}$

Ex: $y''-4y'+5y=0$

$r^2-4r+5=0$

$r=\frac{4\pm\sqrt{16-20}}{2}=2\pm\frac{\sqrt{-4}}{2}$

$=2\pm\frac{\sqrt{4}}{2}\cdot\sqrt{-1}=\underset{\uparrow\,\alpha}{2}\pm\underset{\uparrow\,\beta}{1}\sqrt{-1}$

$y=e^{2t}(C_1\cos t + C_2 \sin t)$

② Linear second order differential equations

Ex: $y'' = 2t \qquad y' = \int 2t\,dt = t^2 + C$

$y' = t^2 + C \qquad y = \int t^2 + C\,dt = \underline{\underline{\frac{1}{3}t^3 + Ct + D}}$

general solution depends on two undetermined coeff.

$\underline{y'' = 2t,\ y(0) = 1,\ y'(0) = 2}$

$y = \frac{1}{3}t^3 + Ct + D$

$$\left.\begin{array}{ll} y(0)=1: & \frac{1}{3}\cdot 0^3 + C\cdot 0 + D = 1 \Rightarrow \underline{D=1} \\ y'(0)=2: & 0^2 + C \overset{!}{=} 2 \qquad \underline{\underline{C=2}} \end{array}\right\} \quad \underline{\underline{y = \frac{1}{3}t^3 + 2t + 1}}$$

Defn: A linear second order differential is a second order differential eqn that can be written

$$\boxed{y'' + a(t)y' + b(t)y = h(t)}$$

It is called homogeneous if $h(t) = 0$, inhomogeneous otherwise. We say that it has constant coeff. if $a(t) = a$ and $b(t) = b$ are constants

Ex: $y'' - 3y' + 2y = 0$ homog.

$y'' - 3y' + 2y = 12e^{-t}$ inhomog.

Explanation: Char. eqn.

$y'' + ay' + by = 0 \longleftrightarrow$ try with $e^{rt} = y$

$y' = e^{rt} \cdot r$

$y'' = e^{rt} \cdot r \cdot r$

$(r^2 e^{rt}) + a \cdot (r e^{rt}) + b(e^{rt}) = 0$

$e^{rt} \cdot (r^2 + ar + b) = 0$

$\boxed{r^2 + ar + b = 0}$ Char. eqn.

Conclusion: $y = e^{rt}$ is a solution of $y'' + ay' + by = 0$

$\Updownarrow$

r is a root in the characteristic eqn.

In case b): See Ex 7.32 in [E]

ii) **The inhomogeneous case:** $y'' + ay' + by = h(t)$

Ex: $\underbrace{y'' - 3y' + 2y} = 12e^{-t}$

$D = \frac{d^2}{dt^2} - 3 \cdot \frac{d}{dt} + 2$

$D(y) = y'' - 3y' + 2y$

$D(y_h + y_p) = D(y_h) + D(y_p)$
$= 0 + h(t) = h(t)$

Superposition principle:

$\boxed{\text{General solution } y = y_h + y_p}$

Works for all linear diff. eqn.s

the general homogeneous solution of $y'' + ay' + by = 0$

a particular solution of $y'' + ay' + by = h(t)$

Ex: $y'' - 3y' + 2y' = 12e^{-t}$

General solution: $Y = Y_h + y_p = \underline{\underbrace{C_1e^{t} + C_2e^{2t}}_{y_h} + \underbrace{2e^{-t}}_{y_p}}$

Y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$\underline{r=1}, \underline{r=2}$

$$Y_h = C_1 \cdot e^{1 \cdot t} + C_2 e^{2t} = \underline{C_1 e^{t} + C_2 e^{2t}}$$

Y_p: $Y'' - 3y' + 2y = \underline{12e^{-t}}$

Guess: $y = \underline{Ae}^{-t}$

$y' = Ae^{-t} \cdot (-1) = \underline{-Ae^{-t}}$

$y'' = -A \cdot e^{-t} \cdot (-1) = \underline{Ae^{-t}}$

$(Ae^{-t}) - 3(-Ae^{-t}) + 2(Ae^{-t}) = 12e^{-t}$

$Ae^{-t} + 3Ae^{-t} + 2Ae^{-t} = 12e^{-t}$

$(\quad 6A \quad)e^{-t} = 12e^{-t}$

Method of undetermined coeff's

- choose a "guess" y(t) such that
  - i) y(t) is of the same form as $\underline{h(t)}$
  - ii) y(t) depends on parameters
- compute $y'(t), y''(t)$ and put it in

Solution if $6A = 12$

$\underline{A = 2} \rightarrow y_p = \underline{2e^{-t}}$

Ex: $y'' - 5y' + 4y = 2e^t$

$$Y = Y_h + Y_p = \underline{\underline{C_1 e^t + C_2 e^{4t} - \frac{2}{3} t e^t}}$$

Y_h: $y'' - 5y' + 4y = 0$

$r^2 - 5r + 4 = 0$

$(r-1)(r-4) = 0 \quad \underline{r_1 = 1}, \underline{r_2 = 4} \Rightarrow y_h = \underline{C_1 e^t + C_2 e^{4t}}$

Y_p: $Y'' - 5y' + 4y = \underline{2e^t}$ $\quad \underbrace{h = 2e^t \quad h' = 2e^t \quad h'' = 2e^t}$

$(Ae^t) - 5(Ae^t) + 4(Ae^t) = 2e^t$

$(\underbrace{A - 5A + 4A}_{0A = 0})e^t = 2e^t$ $\quad$ not possible

$\Leftarrow \begin{cases} \cancel{y = Ae^t} \\ \cancel{y' = Ae^t} \\ \cancel{y'' = Ae^t} \end{cases}$

First guess does not work; multiply with $\underline{t}$

$(2A + \underline{At})e^t - 5(A + \underline{At})e^t + 4(\underline{At})e^t = 2e^t$

$\Leftarrow \begin{cases} Y = \underline{Ate^t} \\ Y' = A \cdot e^t + At \cdot e^t = \underline{(A + At)e^t} \\ Y'' = A \cdot e^t + (A + At) \cdot e^t = \underline{(2A + At)e^t} \end{cases}$

$(\cancel{At - 5At + 4At} + 2A - 5A)e^t = 2e^t$

$-3Ae^t = 2e^t$

Solution if $-3A = 2$

$A = -2/3 \Rightarrow y_p = \underline{-\frac{2}{3} t e^t}$

Ex: $y' - 5y = e^{-t}$ linear first order const. coeff.

$$y = y_h + y_p = \underline{\underline{C\cdot e^{5t} - \frac{1}{6}e^{-t}}}$$

$\underline{y_h}$: $y' - 5y = 0$

$r - 5 = 0$ $r = 5$ $y_h = \underline{C\cdot e^{5t}}$

$\underline{y_p}$: $y' - 5y = e^{-t}$ $\swarrow \begin{cases} y = \underline{A\cdot e^{-t}} \\ y' = -Ae^{-t} \end{cases}$

$$(-Ae^{-t}) - 5(Ae^{-t}) = e^{-t}$$

$$(-6A)e^{-t} = 1\cdot e^{-t}$$

$-6A = 1$

$A = -1/6 \Rightarrow y_p = \underline{-\frac{1}{6}e^{-t}}$