Plan

1. Systems of linear first order differential equations
2. Solution method: The homogeneous case
3. Solution method: The inhomogeneous case
4. Equilibrium states and stability

Plenary Session 4

Mon at 16-19

Lecture 10-12

① Systems of differential equations

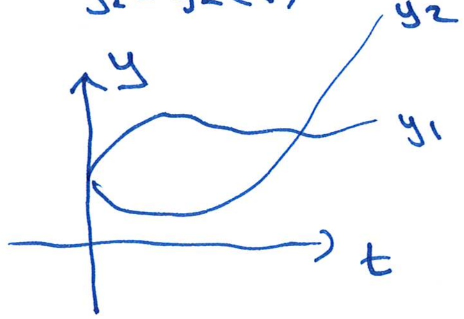
Ex:

$$y_1' = y_1 - 4y_2$$
$$y_2' = -4y_1 + y_2$$

"coupled"

Unknown functions

$$y_1 = y_1(t)$$
$$y_2 = y_2(t)$$



Ex:

$$y_1' = -3y_1$$
$$y_2' = 5y_2$$

"decoupled"

$$y_1' = -3y_1 \underset{\text{sep.}}{\Longrightarrow} \frac{1}{y_1} y_1' = -3 \Rightarrow \dots \Rightarrow \underline{y_1 = C_1 \cdot e^{-3t}}$$

$$y_2' = 5y_2 \underset{\text{sep}}{\Longrightarrow} \quad - - - - \quad \Rightarrow \underline{y_2 = C_2 \cdot e^{5t}}$$

Defn: Systems of linear autonomous first order differential equations:

$$(*) \begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n + b_1 \\ y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n + b_2 \\ \vdots \\ y_n' = a_{n1}y_1 + a_{n2} \cdot y_2 + \dots + a_{nn}y_n + b_n \end{cases}$$

Notation:

$$\underline{y} = (y_1, y_2, \dots, y_n)$$
$$\underline{y}' = (y_1', y_2', \dots, y_n')$$

think of these as column vectors

$$\boxed{\underline{y}' = A \cdot \underline{y} + \underline{b}}$$

$(*)$ in matrix form

Ex: $y_1' = y_1 - 4y_2$
$y_2' = -4y_1 + y_2$

$$\underline{y}' = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} \cdot \underline{y}$$

$\underline{b} = \underline{0}$ (homogeneous)

Note: The system (*) $\underline{y}' = A \cdot \underline{y} + \underline{b}$ of differential equations is $\begin{cases} \text{homogeneous} & \text{if } \underline{b} = \underline{0} \\ \text{inhomogeneous} & \text{if } \underline{b} \neq \underline{0} \end{cases}$

and it is "decoupled" if and only if A is diagonal.

Ex: $\left.\begin{aligned} y_1' &= -3y_1 \\ y_2' &= \quad\quad 5y_2 \end{aligned}\right\} \quad \underline{y}' = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \cdot \underline{y}$

Idea: When A is not diagonal, try to diagonalize it to decouple the system.

Second order differential equations = Special case of a system of two first order linear differential equ.s.

Ex: $y'' - 4y' - 5y = 10$

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned}\right\} \quad \begin{aligned} y_1' &= y' = 0 \cdot y_1 + 1 \cdot y_2 \\ y_2' &= y'' = 10 + 5y_1 + 4y_2 \end{aligned}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

Char. eqn: $r^2 - 4r - 5 = 0$

Eigenvalues of A: $\begin{vmatrix} 0-\lambda & 1 \\ 5 & 4-\lambda \end{vmatrix} = 0$

$$-\lambda(4-\lambda) - 5 = 0$$
$$\underline{\lambda^2 - 4\lambda - 5 = 0}$$

Ex: $y''' - 3y' + 2y = 6$ $\quad r^3 - 3r + 2 = 0$

$$\left.\begin{array}{l} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{array}\right\} \quad \begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y''' = 6 - 2y + 3y' = -2y_1 + 3y_2 + 6 \end{array}$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$$\boxed{\begin{array}{l} A: n \times n\text{-matrix} \\ \underline{y}' = A \cdot \underline{y} \end{array}}$$

② Solution method: Homogeneous case

Ex: $\underline{y}' = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} \cdot \underline{y}$ with $A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(*) \begin{cases} y_1' = y_1 - 4y_2 \\ y_2' = -4y_1 + y_2 \end{cases}$$

We diagonalize $A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix}$:

i) Eigenvalues: $\lambda^2 - 2\lambda - 15 = 0$

$(\lambda - 5)(\lambda + 3) = 0 \qquad \lambda_1 = -3 \qquad \lambda_2 = 5$

ii) Eigenvectors:

$\lambda_1 = -3$: $\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = A - \lambda I$ $\quad 4x - 4y = 0$, y free, $x = y$ $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ base of E_{-3}

$\lambda_2 = 5$: $\begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} = A - \lambda I$

$-4x - 4y = 0$, y free, $x = -y$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ base of E_5

Diagonalization: $P^{-1}AP = D$ with $D = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$ $\quad P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^T$

$\underline{y}' = A\underline{y}$ with $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix}$

Diagonalization: $P^{-1}AP = D$, $D = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$ $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$: $\underline{y} = P \cdot \underline{z} \iff \underline{z} = P^{-1} \cdot \underline{y}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y_1 = z_1 - z_2 \qquad z_1 = \frac{1}{2}(y_1 - y_2)$$
$$y_2 = z_1 + z_2 \qquad z_2 = \frac{1}{2}(y_1 + y_2)$$

$\underline{y}' = A\underline{y}$: $\underline{y}' = (P\underline{z})' = P\underline{z}'$

$P\underline{z}' = A \cdot P\underline{z} \quad | P^{-1} \cdot$

$P^{-1}P\underline{z}' = P^{-1}AP\underline{z} \longrightarrow \underline{z}' = (P^{-1}AP)\cdot\underline{z} = D\underline{z}$

Example: $\underline{z}' = D\underline{z}$ decoupled

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$z_1' = -3z_1 \qquad z_1 = C_1 \cdot e^{-3t}$$
$$z_2' = 5z_2 \qquad z_2 = C_2 \cdot e^{5t}$$

sep. diff. eqn.

Solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} C_1 e^{-3t} \\ C_2 e^{5t} \end{pmatrix}$

$$= \underline{\underline{C_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + C_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{5t}}}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} C_1 e^{-3t} - C_2 e^{5t} \\ C_1 e^{-3t} + C_2 e^{5t} \end{pmatrix}}}$$

Result: Homogeneous case

Let $\underline{y}' = A\underline{y}$ be a system of differential equations, where A is a diagonalizable $n\times n$-matrix. Then the general solution is

$$\underline{y} = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} + \dots + C_n \underline{v}_n e^{\lambda_n t}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A (counted with multiplicity) and $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ are n linearly independent eigenvectors with $\underline{v}_i$ in E_{λ_i}.

Ex: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

upper triangular but not diagonal

$$y_1' = 2y_1 + 3y_2$$
$$y_2' = -y_2$$

$$\underline{y_2 = C_2 \cdot e^{-t}}$$

$$y_1' = 2y_1 + 3C_2 e^{-t}$$
$$\underline{y_1' - 2y_1 = 3C_2 e^{-t}}$$

linear diff. eqn. we can solve

③ Inhomogeneous case: $\underline{y}' = A\underline{y} + \underline{b}$

A: $n \times n$ diagonalizable matrix

$\underline{b}$: n-vector

Ex: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$y_1' = 3y_1 + 4y_2 + 1$
$y_2' = y_1 + 6y_2 + 5$

inhomogeneous

Equilibrium states: $\underline{y}_e$ is an equilibrium state if $\underline{y} = \underline{y}_e$ satisfies: $\begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix} \cdot \underline{y} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix} \cdot \underline{y} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ linear system

Gauss:

$$\left(\begin{array}{cc|c} 3 & 4 & -1 \\ 1 & 6 & -5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 6 & -5 \\ 3 & 4 & -1 \end{array}\right) \xrightarrow{-3} \left(\begin{array}{cc|c} 1 & 6 & -5 \\ 0 & -14 & 14 \end{array}\right)$$

$y_1 + 6y_2 = -5$
$-14y_2 = 14$

$y_1 - 6 = -5$
$y_2 = -1$

$y_1 = 1$
$y_2 = -1$

Eq. state:

$\underline{y}_e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Use the eq. state to transform the system into a homogeneous one:

$\underline{z} = \underline{y} - \underline{y}_e \Rightarrow \underline{z}' = A\underline{z}$ homogeneous

$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 - 1 \\ y_2 + 1 \end{pmatrix}$

Remember:

$\underline{y}' = A\underline{y} + \underline{b}$
$\underline{0} = A\underline{y}_e + \underline{b}$

$\underline{y}' - \underline{0} = A(\underline{y} - \underline{y}_e)$

$\underline{y}' = A \cdot \underline{z}$

$\underline{z}' = A\underline{z}$

Ex: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ $\quad \underline{y}' = A\underline{y} + \underline{b}$ inhomogeneous

i) Find eq. state: $\begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 3 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ linear system
Solve using Gauss

$\underline{y_e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

ii) Transform $\underline{y}' = A\underline{y} + \underline{b}$ into $\boxed{\underline{z}' = A\underline{z}}$ using $\underline{z} = \underline{y} - \underline{y_e}$
homogeneous
$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

iii) Diagonalize A:

a) Eigenvalues: $\lambda^2 - 9\lambda + 14 = 0 \quad \lambda_1 = \underline{2} \quad \lambda_2 = \underline{7}$

b) Eigenvectors:

$\lambda = 2$: $\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} \to \underline{v_1} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$\lambda = 7$: $\begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix} \to \underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

iv) Solution:

$$\underline{z} = C_1 \cdot \underline{v_1} \cdot e^{\lambda_1 t} + C_2\, \underline{v_2}\, e^{\lambda_2 t}$$

$$\underline{z} = C_1 \cdot \begin{pmatrix} 4 \\ -1 \end{pmatrix} e^{2t} + C_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$$

$$\underline{y} = \underline{z} + \underline{y_e} = C_1 \begin{pmatrix} 4 \\ -1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

4 Equilibrium states and stability

i) System of differential equations:

$\underline{y}' = A \cdot \underline{y} + \underline{b}$

Defn: An equilibrium state is a constant vector $\underline{y}_e$ such that

$$\boxed{A \cdot \underline{y}_e + \underline{b} = 0}$$

Stability:

$\underline{y}_e$ is stable if $\underline{y}(0)$ close to $\underline{y}_e$ $\Rightarrow$ $\underline{y}(t) \to \underline{y}_e$ as time passes

unstable — " — $\Rightarrow$ $\underline{y}(t)$ moves away from $\underline{y}_e$ as time passes

Note: General solution (if A is diag.)

$$\underline{y} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \underline{v_2} e^{\lambda_2 t} + \dots + C_n \underline{v_n} e^{\lambda_n t} + \underline{y_e}$$

Result: $\lambda_1, \lambda_2, \dots, \lambda_n < 0 \Rightarrow \underline{y_e}$ stable (and globally as. stable)
otherwise, $\underline{y_e}$ is unstable

ii) Second order differential equation:

$y'' + ay' + by = h$ (a, b, h const.)

To find eq. state / stability, transform the diff. eqn. into a system of linear diff. eqn.

$$\left.\begin{matrix} y_1 = y \\ y_2 = y' \end{matrix}\right\} \quad \left.\begin{matrix} y_1' = y_2 \\ y_2' = -by_1 - ay_2 + h \end{matrix}\right\} \quad \underline{y}' = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \underline{y} + \begin{pmatrix} 0 \\ h \end{pmatrix}$$

Find y_e: $\begin{pmatrix}0 & 1\\ -b & -a\end{pmatrix}\cdot y + \begin{pmatrix}0\\ h\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$ $\qquad -by_1 = -h$

$\left(\begin{array}{cc|c}0 & 1 & 0\\ -b & -a & -h\end{array}\right) \to \left(\begin{array}{cc|c}-b & -a & -h\\ 0 & 1 & 0\end{array}\right)$ $\qquad y_2 = 0$

$b \neq 0$: $\left.\begin{array}{l} y_1 = +h/b \\ y_2 = 0 \end{array}\right\}$ $\underline{y_e} = \begin{pmatrix}+h/b\\ 0\end{pmatrix}$ $\leftarrow y = y_1$

Alternative: $y'' + ay' + by = h \xRightarrow[y''=0]{y'=0} by = h \quad y = \underline{\underline{h/b}}$

Conclusion: $y'' + ay' + by = h$

i) Eq. state: $y_e = h/b$ if $b \neq 0$ (put $y' = y'' = 0$)

ii) Stability: $r_1, r_2 < 0$: y_e is stable

otherwise : y_e is not stable