Plan

1. Linear difference equations
2. Systems of linear first order difference equations
3. Equilibrium states and stability

① Linear difference equations

Ex: $y_{t+1} = y_t + 0.10\cdot y_t - 300$, $y_0 = 5000$

$$y_0 = 5000$$
$$y_1 = 5000 + 0.10\cdot 5000 - 300 = 5200$$

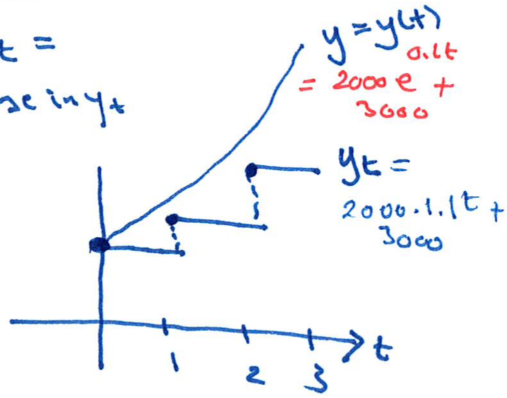
y_0

$$y_2 = 5200 + 0.10\cdot 5200 - 300 = 5420$$
$$\vdots$$
$$y_t = ?$$

Want a closed formula for y_t

Bank Acc. Balance with starting balance 5000, interest 10%, withdr. 300

$$y_{t+1} = 1.1\,y_t - 300 \iff \underbrace{y_{t+1} - y_t}_{\Delta y_t = \text{ change in } y_t} = 0.10\,y_t - 300$$



Note: The solutions of difference equations are sequences

$$y_t = \{y_0, y_1, y_2, \dots\}$$

Differential eqn: $y' - 0.10y = -300$

$$y = y_h + y_p = C\cdot e^{0.10t} + 3000$$
$$= 2000\,e^{0.10t} + 3000$$

$(e^{0.1} = 1{,}105\ldots)$

Ex: $y_{t+1} - y_t = 0{,}6\, y_t - 300$

$y_{t+1} = 1{,}1 y_t - 300$

$$\boxed{y_{t+1} - 1.1 y_t = -300}$$

std. form of a first order linear difference equation

$y_{t+1} - a y_t = h_t$

Compare with:

$y' - ay = h(t)$

Superposition principle:

$$y_t = y_t^h + y_t^p = \underline{C \cdot 1{,}1^t + 3000}$$

y_t^h: $y_{t+1} - 1.1 y_t = 0$

Char. eqn: $r - 1.1 = 0$

$r = 1.1 \rightarrow y_t^h = \underline{\underline{C \cdot 1.1^t}}$

using $\underline{\underline{r^t}}$ instead of $\underline{e^{rt}}$

$y_1 = 1.1 y_0$

$y_2 = 1.1 y_1 = 1.1^2 y_0$

$y_3 = 1.1 y_2 = 1.1^3 y_0$

y_t^p: $y_{t+1} - 1.1 y_t = -300$

Guess: $y_t = A \leftarrow \{A, A, A, \ldots\}$

$$A - 1.1 A = -300$$

$$\frac{-0.1 A}{-0.1} = \frac{-300}{-0.1}$$

$$A = \underline{3000} \rightarrow y_t^p = \underline{\underline{3000}}$$

General solution: $y_t = \underline{C \cdot 1.1^t + 3000}$

Initial condition: $y_0 = 5000$: $y_0 = C \cdot 1.1^0 + 3000 = 5000$

$C = \underline{2000}$

Solution of difference equation with initial condition: $y_t = \underline{\underline{2000 \cdot 1.1^t + 3000}}$

First order linear: $y_{t+1} + a y_t = h_t$

Second order linear: $y_{t+2} + a y_{t+1} + b y_t = h_t$

linear difference equations

$y_{t+1} - y_t = \Delta y_t$: change (replaces y')

$\Delta y_{t+1} - \Delta y_t = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$

$= y_{t+2} - 2y_{t+1} + y_t$ (replaces y'')

Ex: $y_{t+1} = 1.10 y_t - 300 - 10t$, $y_0 = 5000$

difference eqn. first order y_t, y_{t+1} linear

$y_{t+1} - 1.10 y_t = -300 - 10t$

$C = 1000$: $y_t = 1000 \cdot 1.1^t + 100t + 4000$

$y_t = y_t^h + y_t^p = C \cdot 1.10^t + 100t + 4000$

y_t^h: $y_{t+1} - 1.10 y_t = 0$

$r - 1.10 = 0 \quad r = 1.10 \quad y_t^h = C \cdot 1.1^t$

y_t^p: $y_{t+1} - 1.10 y_t = -300 - 10t$

$y_t = At + B \longrightarrow y_t^p = 100t + 4000$

$y_{t+1} = A(t+1) + B = At + A + B$

$\downarrow$

$(At + A + B) - 1.10(At + B) = -300 - 10t$

$\underbrace{(A + B - 1.10B)}_{= -300} + \underbrace{(A - 1.1A)}_{= -10} t = -300 - 10t$

$\dfrac{-0.1A}{-0.1} = \dfrac{-10}{-0.1}$

$A = 100$

$100 - 0.1B = -300$

$-0.1B = -400$

$B = 4000$

Ex: $y_{t+2} = y_{t+1} + y_t$, $y_0 = 1$, $y_1 = 1$ $\leadsto$ Fibonacci sequence

1, 1, 2, 3, 5, 8, ..

$\Updownarrow$

$y_{t+2} - y_{t+1} - y_t = 0$ homogeneous

$$y_t = y_t^h = \underline{C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t}$$

y_t^h: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} = r_1, r_2$$

$y_0 = 1$: $C_1 \cdot 1 + C_2 \cdot 1 = 1$ $\quad C_1 + C_2 = 1$

$y_1 = 1$: $C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$

Result: $y_{t+2} + a y_{t+1} + b y_t = 0$

Char. eqn: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

i) $a^2 - 4b > 0$: $r_1 \neq r_2$ $\Rightarrow y_t = \underline{C_1 \cdot r_1^t + C_2 \cdot r_2^t}$

ii) $a^2 - 4b = 0$: $r_1 = r_2 = -a/2$

$$\Rightarrow y_t = \underline{C_1 \cdot (-a/2)^t + C_2 t (-a/2)^t}$$

Ex: Final exam 01/2021, Q2a):

$$y_{t+2} - y_{t+1} - 2y_t = 4t$$

(second order linear difference eqn.)

$$y_t = y_t^h + y_t^p = \underline{\underline{c_1 \cdot 2^t + c_2(-1)^t - 2t - 1}}$$

y_t^h:

$$y_{t+2} - y_{t+1} - 2y_t = 0$$

$$r^2 - r - 2 = 0$$

$$r = \frac{1 \pm \sqrt{(-1)^2 - 4(-2)}}{2} = \frac{1 \pm \sqrt{9}}{2} = \underline{2, -1}$$

$$y_t^h = \underline{c_1 \cdot 2^t + c_2 \cdot (-1)^t}$$

y_t^p:

$$y_{t+2} - y_{t+1} - 2y_t = 4t$$

$$\begin{cases} y_t = \underline{At + B} \\ y_{t+1} = A(t+1) + B = \underline{At + A + B} \\ y_{t+2} = A(t+2) + B = \underline{At + 2A + B} \end{cases}$$

$$(At + 2A + B) - (At + A + B) - 2(At + B) = 4t$$

$$(\underbrace{-2A}_{=4})t + (\underbrace{A - 2B}_{=0}) = 4t$$

$\underline{A = -2}$ $\quad -2 - 2B = 0$, $\underline{B = -1}$ $\quad y_t^p = \underline{-2t - 1}$

$\underline{y_0 = 1}$: $\quad c_1 + c_2 - 1 = 1$

$\underline{y_1 = 1}$: $\quad 2c_1 - c_2 - 3 = 1$

$$\boxed{\begin{aligned} c_1 + c_2 &= 2 \\ 2c_1 - c_2 &= 4 \end{aligned}}$$

$3c_1 = 6 \quad \underline{c_1 = 2} \quad \underline{c_2 = 0}$

$$\Rightarrow y_t = 2 \cdot 2^t + 0 \cdot (-1)^t - 2t - 1 = \underline{\underline{2^{t+1} - 2t - 1}}$$

2 Systems of first order linear difference equ's

$$y_{t+1} = 0.85y_t + 0.14z_t$$
$$z_{t+1} = 0.15y_t + 0.86z_t$$

$$\begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 0.85 & 0.14 \\ 0.15 & 0.86 \end{pmatrix} \begin{pmatrix} y_t \\ z_t \end{pmatrix} \Leftrightarrow \underline{y}_{t+1} = A\underline{y}_t$$

General solution: $\boxed{\begin{pmatrix} y_t \\ z_t \end{pmatrix} = c_1 \cdot \underline{v_1}\lambda_1^t + c_2\underline{v_2}\cdot\lambda_2^t}$

(when A is diagonalizable,
λ_1, λ_2 eigenvalues of A, $\underline{v_1}, \underline{v_2}$ eigenvectors)

$$A = \begin{pmatrix} 0.85 & 0.14 \\ 0.15 & 0.86 \end{pmatrix}$$

i) Eigenval: $\begin{vmatrix} 0.85-\lambda & 0.14 \\ 0.15 & 0.86-\lambda \end{vmatrix} = \lambda^2 - 1.71\lambda + 0.71 = 0$

$$\lambda = \frac{1.71 \pm \sqrt{1.71^2 - 4\cdot 0.71}}{2}$$

$\underline{\lambda = 1}$ or $\lambda = \underline{0.71}$

ii) Eigenvect:

$\lambda = 1$: $\begin{pmatrix} -0.15 & 0.14 \\ 0.15 & -0.14 \end{pmatrix} \rightsquigarrow \underline{v_1} = \begin{pmatrix} 14 \\ 15 \end{pmatrix}$

$\lambda = 0.71$: $\begin{pmatrix} 0.14 & 0.14 \\ 0.15 & 0.15 \end{pmatrix}$ $\quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} y_t \\ z_t \end{pmatrix} = c_1 \cdot \begin{pmatrix} 14 \\ 15 \end{pmatrix} \cdot 1^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 0.71^t = \underline{\underline{\begin{pmatrix} 14c_1 - c_2\cdot 0.71^t \\ 15c_1 + c_2 \cdot 0.71^t \end{pmatrix}}}$$

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3 Equilibrium states and stability for difference eqn.

Ex: $y_{t+1} - 1.1\, y_t = -300$

first order difference eqn.
autonomous
$y_{t+1} = F(y_t)$

Eq. state: $y_{t+1} - y_t = 0.1 y_t - 300$

$$y_{t+1} - y_t = 0.1\, y_t - 300$$

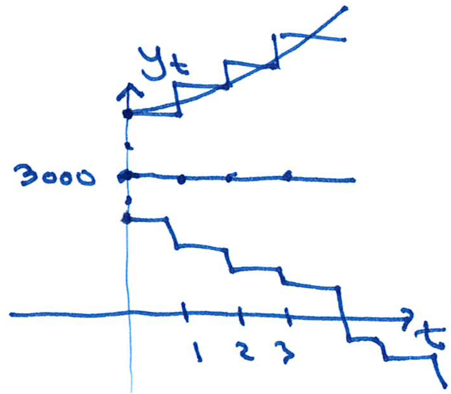
$$\Vert$$

$$0.1 y_t - 300 = 0$$

$$0.1\, y_t = 300$$

$$y_t = \underline{3000}$$

$$y_e = \underline{\underline{3000}}$$



General solution: $y_t = \underline{C \cdot 1.1^t + 3000}$

$\Rightarrow y_e = \underline{3000}$ is unstable

In general: $y_{t+1} = F(y_t)$

$$y_{t+1} - y_t = F(y_t) - y_t$$

$$y_{t+1} - a y_t = h_t$$

Eq. state:
$F(y_t) - y_t = 0$

Stability:
$-1 < a < +1$: y_e is stable
otherwise : y_e is unstable

System: $\underline{y}_{t+1} = A \cdot \underline{y}_t$

$\underline{y}_t = \begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{pmatrix} \quad \underline{y}_{t+1} = \begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \\ y_{3,t+1} \end{pmatrix}$

Eq. state: $\underline{y}_{t+1} - \underline{y}_t = A \cdot \underline{y}_t - \underline{y}_t = (A - I)\,\underline{y}_t$

$\underline{y}_t$ is an eq. state $\Leftrightarrow$ $\underline{y}_t$ is in E_1, an eigenvector for $\underline{\lambda = 1}$

Stability: Formula for the solution

$\underline{y}_t = c_1 \cdot \underline{v}_1 \cdot \lambda_1^t + c_2 \cdot \underline{v}_2 \cdot \lambda_2^t + \ldots$

$\underline{y}_e$ stable $\Leftrightarrow$ $-1 < \lambda_1, \lambda_2, \ldots, \lambda_n < 1$