Plan
1 Review of the course
2 Final exam 12/2021

Remember: $\boxed{\text{Course evaluation}}$

How to prepare for the exam:

- do exam problems (2017-2022)
- look at Lecture 14 from last y.

Monday: Exercise session
D1-065 from 1600-1800

① Review

- Linear algebra / matrix methods
- optimization problems (unconstr. / Lagrange / KT)
- difference / differential equations

- more focus on:
  matrix methods
  inner products

- not on last year's exam but important:
  - unconstrained opt.
  - envelope thms
  - computation of eigenvalues / diagonalization
  - exact diff. eqns,
  - systems of difference equations
  - eq. states and stability

2 Exam 12/2021

1. $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ 4 & 2 & 2 & 0 \\ 1 & -2 & 8 & 5 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 2 \\ -3 \end{pmatrix}$

a) $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ 4 & 2 & 2 & 0 \\ 1 & -2 & 8 & 5 \end{pmatrix} \begin{matrix} -3 \\ -4 \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -6 & -4 \\ 0 & 2 & -6 & -4 \\ 0 & -2 & 6 & 4 \end{pmatrix} \begin{matrix} -1 \\ 1 \end{matrix}$

$\to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -6 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E$

$\mathrm{rk}(A) = 2$

Base of Col(A): $\left\{ \underline{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \end{pmatrix} \right\}$

b) $A\underline{v} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ 4 & 2 & 2 & 0 \\ 1 & -2 & 8 & 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \cdot \underline{v} = \begin{pmatrix} -\lambda \\ 0 \\ 2\lambda \\ -3\lambda \end{pmatrix}$

holds if $\lambda = 0$

$\Rightarrow$ $\underline{v}$ is an eigenvector of A with eigenvalue $\lambda = 0$

c) $|A| = |E| = 1 \cdot 2 \cdot 0 \cdot 0 = 0$ $\quad$ ($|A| = 0$ since $\mathrm{rk}\, A = 2 < 4$)

d) S: symm. 3×3-matrix with eigenvalues $\lambda = 1, 2, 4$

i) $|S| = 1 \cdot 2 \cdot 4 = 8 \neq 0$

$\Rightarrow$ S is invertible.

ii) Definiteness of S^{-1}:

$\to$ Eigenvalues of S^{-1}: $\frac{1}{1} = 1, \frac{1}{2}, \frac{1}{4} > 0$

$\Rightarrow$ S^{-1} positive definite

Note: $S\underline{v} = \lambda \underline{v}$

$S^{-1}S\underline{v} = S^{-1}\lambda\underline{v} = \lambda S^{-1}\underline{v}$

$\underline{v} = \lambda S^{-1}\underline{v} \Rightarrow \frac{1}{\lambda}\underline{v} = S^{-1}\underline{v}$

Alt: S is diagonalizable: $P^{-1}SP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$S = P \cdot D \cdot P^{-1}$

$\Leftarrow S = P \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} P^{-1}$

$$S^{-1} = (PDP^{-1})^{-1}$$
$$= (P^{-1})^{-1} \cdot D^{-1} \cdot P^{-1}$$
$$= P \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} P^{-1}$$

$D^{-1} = \begin{pmatrix} 1/1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$

$\Rightarrow$ S^{-1} is positive defn.

2. a) $y'' + y' = 6e^{3t}$ second order linear

$$y = y_h + y_p = C_1 + C_2 e^{-t} + \frac{1}{2} e^{3t}$$

y_h: $y'' + y' = 0$

$r^2 + r = 0$

$r(r+1) = 0$

$r = 0,\ r = -1$

$y_h = C_1 e^{0t} + C_2 e^{-t}$

$= C_1 + C_2 e^{-t}$

$h = 6e^{3t}$

$h' = 18e^{3t}$

$h'' = 54e^{3t}$

y_p: $y'' + y' = 6e^{3t}$

$y = Ae^{3t}$

$y' = Ae^{3t} \cdot 3$

$y'' = Ae^{3t} \cdot 3 \cdot 3$

$9Ae^{3t} + 3Ae^{3t} = 6e^{3t}$

$12A \cdot e^{3t} = 6 \cdot e^{3t}$

$12A = 6 \quad A = 6/12 = 1/2$

$y_p = \frac{1}{2} e^{3t}$

b) $t(y'-y)=y \quad | : t$

$y'-y=\frac{y}{t} \quad | +y$

$y'=\frac{y}{t}+y=y\left(\frac{1}{t}+1\right) \quad | : y$ sep. first order

$y'=\left(1+\frac{1}{t}\right)y$ lin -"-

sep.

$$\frac{1}{y}y'=\frac{1}{t}+1$$

$$\int \frac{1}{y}\,dy=\int \frac{1}{t}+1\,dt$$

$$\ln|y| = \ln|t|+t+C \quad | \exp$$

$$e^{\ln|y|}=e^{\ln|t|+t+C}$$

$$|y|=e^{\ln|t|}\cdot e^{t}\cdot e^{C}$$

$$|y|=|t|\cdot e^{t}\cdot e^{C}$$

$$y=\pm t e^{t}\cdot e^{C}=\underline{\underline{Kte^{t}}}$$

c) $y_{t+2}+3y_{t+1}-4y_t=5$ lin. second order difference eqn.

$$y_t=y_t^h+y_t^p=\underline{\underline{C_1\cdot(-4)^t+C_2+t}}$$

y_t^h: $y_{t+2}+3y_{t+1}-4y_t=0$ $\qquad y_t^h=C_1\cdot(-4)^t+C_2\cdot 1^t$

$r^2+3r-4=0$ $\qquad =\underline{C_1\cdot(-4)^t+C_2}$

$(r+4)(r-1)=0$

$\underline{r=-4}, \underline{r=1}$

y_t^p: $y_{t+2} + 3y_{t+1} - 4y_t = 5$

$\cancel{A + 3A - 4A = 5}$

$\cancel{0 \cdot A = 5}$

~~no solution~~

$\cancel{y_t = A}$ $\quad$ $\boxed{y_t = At}$

$\cancel{y_{t+1} = A}$ $\quad$ $y_{t+1} = A(t+1)$

$\cancel{y_{t+2} = A}$ $\quad$ $y_{t+2} = A(t+2)$

$$(At + 2A) + 3(At + A) - 4(At) = 5$$

$$\cancel{At + 3At - 4At} + 2A + 3A = 5$$

$$5A = 5$$

$$\underline{A = 1} \quad \to \quad y_t^p = \underline{t}$$

d) $\underline{y}' = A\underline{y} \Rightarrow \underline{y} = \underline{c_1 \cdot v_1 \cdot e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + c_3 \underline{v_3} e^{\lambda_3 t}}$

Eigenvalues:
$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 0-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\cdot(-\lambda(1-\lambda)+1) - 1\,(1-\lambda+2) = 0$$

$$(3-\lambda)\cdot(\lambda^2 - \lambda + 1) - 1\cdot(3-\lambda) = 0$$

$$(3-\lambda)\cdot(\lambda^2 - \lambda \cancel{+1} \cancel{-1}) = (3-\lambda)\lambda(\lambda-1) = 0$$

$$\underline{\lambda_1 = 3},\ \underline{\lambda_2 = 0},\ \underline{\lambda_3 = 1}$$

Eigenvectors:

$\underline{\lambda = 3}$: $\begin{pmatrix} -2 & 1 & 2 \\ -1 & -3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v_1} = \underline{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

$\underline{\lambda = 0}$: $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{aligned} -x + z &= 0 \\ y + 3z &= 0 \end{aligned}$ $\quad \underline{v_2} = \underline{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}}$

$y = -3z$
$x = z$ $\quad \begin{pmatrix} z \\ -3z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

$\lambda = 1$: $\begin{pmatrix} 0 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\quad -x - y + z = 0 \quad x = 3z \quad v_3 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$\quad y + 2z = 0 \quad y = -2z$

$$\begin{pmatrix} 3z \\ -2z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{y} = c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_2 \cdot \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} e^{t}$$

$$\mathbf{y}(0) = c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & -3 & -2 & -5 \\ 1 & 1 & 1 & 3 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & -2 & -2 \end{array}\right) \qquad \begin{array}{l} c_1 = 1 \\ c_2 = 1 \\ c_3 = 1 \end{array}$$

$$\mathbf{y}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} e^{t}$$

3. max $f = x + y + z + w$ when $g(x,y,z,w) \le 18$

a) $A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$ symmetric matrix of g

$D_1 = 3 \qquad D_3 = 4 \cdot (-2) - 2 \cdot 2 + 7 \cdot 2 = 2$

$D_2 = 2 \qquad D_4 = 4 \cdot D_3 + 1 \cdot \begin{vmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \\ 4 & 7 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ 2 & 7 & 0 \end{vmatrix}$

$$= 8 + (4 \cdot 6 - 7 \cdot 4) + (2 \cdot 6 - 7 \cdot 2)$$

$$= 8 - 4 - 2 = 2$$

g is positive defin.

KT-problem in std. form: max $f = x+y+z+w$ when $g = 3x^2 + \dots + 4w^2 \leq 18$

$f = B\underline{x}$ $\quad g = \underline{x}^T A \underline{x} \leq 18$

$B = (1\ 1\ 1\ 1)$ $\quad A$ is as in (a).

b) $\mathcal{L} = x+y+z+w - \lambda(3x^2 + \dots + 4w^2 - 18)$

$= B\underline{x} - \lambda \cdot (\underline{x}^T A \underline{x} - 18)$

FOC: $\mathcal{L}'(\underline{x}) = B^T - \lambda(2A\underline{x}) = \underline{0}$

C: $\underline{x}^T A \underline{x} \leq 18$

CSC: $\lambda \geq 0,\ \lambda \cdot (\underline{x}^T A \underline{x} - 18) = 0$

Alt: FOC: $\mathcal{L}'_x = 1 - \lambda \cdot (\underbrace{6x + \dots}_{g'_x}) = 0$

$\mathcal{L}'_y = 1 - \lambda(\underbrace{\dots}_{g'_y}) = 0$

$\mathcal{L}'_z = 1 - \lambda(\underbrace{\dots}_{g'_z}) = 0$

$\mathcal{L}'_w = 1 - \lambda(\underbrace{\dots}_{g'_w}) = 0$

C: $\underbrace{3x^2 + \dots}_{g} \leq 18$

CSC: $\lambda \geq 0,\ \lambda(\underbrace{3x^2 + \dots - 18}_{g-18}) = 0$

c) NDCQ: i) $g = 18$ (binding): $\text{rk}\,(g'_x\ g'_y\ g'_z\ g'_w) = 1$

ii) $g < 18$ (non-binding): no condition

Adm. pts where NDCQ fails:

ii) no pts. $\quad$ i) $g'_x = g'_y = g'_z = g'_w = 0$ and $g(x,y,z,w) = 18$

$g(\underline{x}) = \underline{x}^T A \underline{x}$ $\quad A\underline{x} = \underline{0}$ $\quad$ since A is

$g'(\underline{x}) = 2A\underline{x} = \underline{0}$ $\quad$ 1:2 $\quad \underline{x} = \underline{0}$ $\quad$ invertible $(D_4 = 2 \neq 0)$

Stat pts of g: $(0,0,0,0)$
but $g(0,0,0,0) = 0 \neq 18$
$\Rightarrow$ no adm pts where NDCQ fails

Alt: $g'_x = 6x + 2y + 8z - 2w = 0$
$g'_y = 2x + 2y + 4z + 2w = 0$
$g'_z = \text{----} = 0$
$g'_w = \text{----} = 0$

d) Solve the KT problem

KT-cond:
FOC $B^T - \lambda \cdot 2A\underline{x} = \underline{0}$
C $\underline{x}^T A \underline{x} \leq 18$
CSC $\lambda \geq 0,\ \lambda(\underline{x}^T A \underline{x} - 18) = 0$

i) Binding case: $g = 18$

ii) Non-binding case: $g < 18$

$\lambda = 0 \Rightarrow B^T - \underline{0} = \underline{0}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ impossible

i) Binding case: $g = \underline{x}^T A \underline{x} = 18,\ \lambda \geq 0 \Rightarrow \lambda > 0$

$B^T - 2\lambda A\underline{x} = 0$

$B^T = 2\lambda A\underline{x} \quad | : 2\lambda$

$A\underline{x} = \frac{1}{2\lambda} B^T = \underbrace{\frac{1}{2\lambda}}_{t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 3 & 1 & 4 & -1 & t \\ 1 & 1 & 2 & 1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \xrightarrow{} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -t \\ 1 & 1 & 2 & 1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -t \\ 0 & 2 & 2 & 4 & 2t \\ 0 & 6 & 7 & 12 & 5t \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \cdot(-3) \to \left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -t \\ 0 & 2 & 2 & 4 & 2t \\ 0 & 0 & 1 & 0 & -t \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$x = -t + 2t = \underline{t} \quad \Leftarrow \quad x - y \quad -3w = -t$$

$$2y = 2t - 2(-t) = 4t \Leftarrow 2y + 2z + 4w = 2t$$

$$y = \underline{2t} \qquad z = \underline{-t}$$

$$w = \underline{0}$$

$$\Rightarrow (x,y,z,w) = \underline{(t, 2t, -t, 0)}$$

$\underline{c:}$ $g(t, 2t, -t, 0) = 18: \; 3t^2 + 4t^2 - 8t^2 + 4t^2 - 8t^2 + 7t^2 = 18$

$$2t^2 = 18$$

$$t^2 = 9$$

$$t = \pm 3$$

$$t = 3$$

$t = \frac{1}{2\lambda} = 3$ or $t = \frac{1}{2\lambda} = -3$

$\lambda = \underline{1/6} > 0$ $\quad \cancel{\lambda = -\frac{1}{6}}$

$\Rightarrow$ $\underline{\text{cand pts:}}$ $(x,y,z,w;\lambda) = \underline{(3,6,-3,0;1/6)}$ $\quad f = 6$

$\underline{\text{SOC:}}$ test if this cand. pt. is max

$$h = L(x,y,z,w;1/6) = x+y+z+w - \frac{1}{6}(3x^2 + \ldots - 18)$$

$$= B\underline{x} - \frac{1}{6}(\underline{x}^T A \underline{x} - 18)$$

$$H(h) = 0 - \frac{1}{6}(2A) = -\frac{1}{3}A \quad \underline{\text{neg. defn.}}$$

since A is pos. defn. from (a)

$\Downarrow$ $\underline{\text{SOC}}$

Since h is $\underline{\text{concave}}$, $f_{max} = f(3,6,-3,0) = \underline{\underline{6}}$