① Quadratic functions and optimization

Ex: max/min $f(x,y,z) = 4x^2 - 4xy + 2y^2 + 4xz + 2z^2 - 4x - 4z + 3$

$$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + 3$$

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \qquad B = (-4 \;\; 0 \;\; -4)$$

Theory:

$$\boxed{\begin{aligned} f(\underline{x}) &= \underline{x}^T A \underline{x} + B\underline{x} + C \\ \Rightarrow f'(\underline{x}) &= 2A\cdot\underline{x} + B^T \\ f''(\underline{x}) &= 2A \end{aligned}}$$

Stationary pts: $2A\underline{x} + B^T = \underline{0}$

Hessian: $H(f) = 2A$

Stationary pts: $2A\underline{x} + B^T = 0 \Rightarrow 2A\underline{x} = -B^T$

$A\underline{x} = -\frac{1}{2}B^T$

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \qquad \begin{aligned} D_1 &= 4 \\ D_2 &= 4 \\ D_3 &= 2(-4) + 2\cdot 4 = 0 \end{aligned}$$

positive semidefinite (by the PRC), and not invertible

$$\left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ -2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 2 \end{array}\right) \begin{matrix} \\ ]\ 1/2 \\ ]\ -1/2 \end{matrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 4 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ \cancel{0} & \cancel{1} & \cancel{1} & \cancel{1} \end{array}\right)]\ -1$$

z free

$y + z = 1 \Rightarrow y = \underline{1 - z}$

$4x - 2(1-z) + 2z = 2$

$4x = -4z + 4$

$x = \underline{1 - z}$

$H(f) = 2A$ positive semidef. for all $\underline{x}$

$\Downarrow$

f convex $\Longrightarrow$ global min: $(x,y,z) = (1-z, 1-z, z)$ with z free

Conclusion: $f_{min} = f(1,1,0) = f(0,0,1) = \underline{\underline{1}}$ at $\underline{(x,y,z) = (1-z, 1-z, z)}$ with z free
$(z=0)$ $(z=1)$

Fact: If there are several global max/min pts, they have the same value.

Ex: max/min $f(x,y,z) = x^2 + 2xy + 4y^2 + z^2$ when $x+y+z=1$

$= \underline{x}^T A \underline{x}$ $\qquad \mathbf{B} \cdot \underline{x} = 1$

$D_1 = 1$
$D_2 = 3$
$D_3 = 3$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ positive defn. $\qquad B = (1\ 1\ 1)$

$L = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 1)$

FOC: $L'(\underline{x}) = \boxed{2A\underline{x} - \lambda(B^T) = \underline{0}}$

C: $\boxed{B\underline{x} = 1}$

FOC: $2A\underline{x} = \lambda B^T$

$A\underline{x} = \frac{\lambda}{2} B^T$ $\qquad$ A is invertible

$A^{-1}A\underline{x} = A^{-1}(\frac{\lambda}{2}B^T)$

$\underline{x} = \frac{\lambda}{2}(A^{-1} \cdot B^T)$

C: $B\underline{x} = 1$

$B \cdot \frac{\lambda}{2}(A^{-1}B^T) = 1$

$\frac{\lambda}{2} \underbrace{BA^{-1}B^T} = 1$

$\lambda \cdot \underbrace{(BA^{-1}B^T)}_{>0} = 2$

$\lambda = \dfrac{2}{BA^{-1}B^T}$

$\underline{x} = \left(\dfrac{1}{BA^{-1}B^T}\right) \cdot A^{-1}B^T$

SOC $\Rightarrow$ $\underline{\underline{\min}}$

Theory:
A pos. defn. $\Rightarrow$ A^{-1} pos. defn.
A neg. defn. $\Rightarrow$ A^{-1} neg. defn.

A pos. defn. $\Leftrightarrow$ $\lambda_1, \lambda_2, \dots, \lambda_n > 0$

$\Rightarrow |A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n > 0$ $\quad A^{-1}$ exists

$\underline{\lambda \text{ eigenvalue of } A}$:

$A\underline{x} = \lambda \underline{x}$ $\qquad A^{-1}A\underline{x} = A^{-1}\lambda\underline{x}$

$\underline{x} = \lambda \cdot A^{-1}\underline{x}$

$\frac{1}{\lambda}$ is eigenvalue of A^{-1} $\Leftarrow$ $\frac{1}{\lambda} \cdot \underline{x} = A^{-1} \cdot \underline{x}$

$\frac{1}{\lambda} = \lambda^{-1}$

Eigenvalues of A^{-1}: $1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_n > 0$

max/min $f(\underline{x}) = (x\ y\ z)\begin{pmatrix}1&1&0\\1&4&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$ when $(1\ 1\ 1)\begin{pmatrix}x\\y\\z\end{pmatrix} = 1$

$= \underline{x}^T A \underline{x}$ $\quad \underline{A = \begin{pmatrix}1&1&0\\1&4&0\\0&0&1\end{pmatrix}}$ $\quad B\underline{x} = 1$

$\underline{B = (1\ 1\ 1)}$

$\mathcal{L} = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 1)$

$\underline{\text{FOC}}$: $\mathcal{L}'(\underline{x}) = \boxed{\begin{array}{r} 2A\underline{x} - \lambda B^T = \underline{0} \\ B\cdot \underline{x} = 1 \end{array}}$

$\underline{\text{C}}$:

$A^{-1} = \frac{1}{3}\begin{pmatrix}4&-1&0\\-1&1&0\\0&0&3\end{pmatrix}^T = \frac{1}{3}\begin{pmatrix}4&-1&0\\-1&1&0\\0&0&3\end{pmatrix}$

$2A\underline{x} = \lambda B^T$

$A\underline{x} = \frac{\lambda}{2}\cdot B^T$ $\quad \begin{pmatrix}1&1&0\\1&4&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{\lambda}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$\underline{x} = A^{-1}\cdot\frac{\lambda}{2}\cdot B^T$

$= \frac{\lambda}{2} A^{-1}\cdot B^T$ $\quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{\lambda}{2}\cdot\frac{1}{3}\begin{pmatrix}4&-1&0\\-1&1&0\\0&0&3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$

$= \frac{\lambda}{6}\begin{pmatrix}3\\0\\3\end{pmatrix} = \underline{\frac{\lambda}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}} = \begin{pmatrix}\lambda/2\\0\\\lambda/2\end{pmatrix}$

$\underline{\text{FOC+C}}$:

$(1/2, 0, 1/2; 1)$ $\quad f = \frac{1}{4} + 2\cdot 0 + 4\cdot 0 + 1/4 = \underline{1/2}$

C: $x + y + z = \lambda/2 + 0 + \lambda/2 = 1$

$\underline{\lambda = 1}$

$(x, y, z) = (1/2, 0, 1/2)$

$\underline{\text{SOC}}$: $h(x,y,z) = \mathcal{L}(x,y,z;1) = \underline{x}^T A \underline{x} - 1\cdot(x+y+z-1)$

$H(h) = 2A$ positive def. $\Rightarrow h$ convex $\xrightarrow{\text{SOC}}$ $\underline{(1/2, 0, 1/2)}$ is $\underline{\text{min}}$

$f_{\min} = f(1/2, 0, 1/2)$

$= \underline{\underline{1/2}}$

Ex: max/min $f(x,y,z) = x^2 + 2xy + 4y^2 + z^2$ when $x^2 + y^2 + z^2 = 6$

$$= \underline{x}^T A \underline{x} \qquad\qquad \underline{x}^T I \underline{x} = 6$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T I \underline{x} - 6)$$

FOC: $L'(\underline{x}) = \boxed{\begin{matrix} 2A\underline{x} - \lambda(2I\underline{x}) = \underline{0} \\ \underline{x}^T I \underline{x} = 6 \end{matrix}}$

C:

$2A\underline{x} - \lambda 2I\underline{x} = \underline{0} \quad |:2$

$A\underline{x} - \lambda I \underline{x} = 0$

$\boxed{(A - \lambda I)\underline{x} = \underline{0}}$

$\underline{x} = \underline{0}$ or $|A - \lambda I| = 0$

$(x,y,z) = (0,0,0)$

$x^2 + y^2 + z^2 = 0 \neq 6$

no cand. pts

λ is an eigenvalue of A and $\underline{x}$ is in E_λ

C: $x^2 + y^2 + z^2 = 6$

$\|\underline{x}\| = \sqrt{6}$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

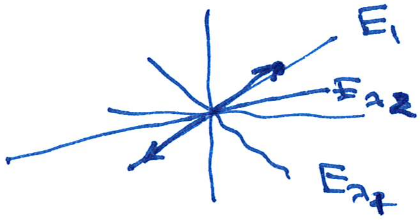
$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot(\lambda^2 - 5\lambda + 3) = 0$$

$\lambda = 1$ or $\lambda = \dfrac{5 \pm \sqrt{25-12}}{2}$

$\lambda_1 = 1 \qquad \dim E_1 = 1$

$\lambda_2 = \dfrac{5+\sqrt{13}}{2} \qquad \dim E_{\lambda_2} = 1$

$\lambda_3 = \dfrac{5-\sqrt{13}}{2} \qquad \dim E_{\lambda_3} = 1$



six candidate pts.

$\underline{x}$ in E_1: $f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T(1\cdot\underline{x}) = 1\cdot \underline{x}^T\underline{x} = 1\cdot(x^2+y^2+z^2) = 1\cdot 6 = 6$

$\underline{x}$ in E_{λ_2}: $f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T(\lambda_2 \underline{x}) = \lambda_2 \underline{x}^T\underline{x} = 6\lambda_2 > 24 \leftarrow$ max

$\underline{x}$ in E_{λ_3}: $f(\underline{x}) = \ldots = 6\cdot\lambda_3 < 6 \leftarrow$ min

$f_{max} = 6\cdot\left(\dfrac{5+\sqrt{13}}{2}\right) = \underline{3(5+\sqrt{13})} \simeq 25.8 \qquad f_{min} = 6\left(\dfrac{5-\sqrt{13}}{2}\right) = \underline{3(5-\sqrt{13})} \simeq 4.2$

SOC:

$\lambda_2 = \frac{5+\sqrt{13}}{2}$: two cand. pts with $f = 6\lambda_2 \simeq \underline{25.8}$

$$h(x,y,z) = L(x,y,z;\lambda_2) = \underline{x}^T A \underline{x} - \lambda_2(\underline{x}^T I \underline{x} - 6)$$
$$= \underline{x}^T (A-\lambda_2 I)\underline{x} + 6\lambda_2$$

$H(h) = 2(A-\lambda_2 I)$

A has eigenvalues $1, \lambda_2, \lambda_3$

$A-\lambda_2 I$ — " — $\underset{<0}{1-\lambda_2}, \underset{=0}{\lambda_2-\lambda_2}, \underset{<0}{\lambda_3-\lambda_2}$

$\Rightarrow h$ concave

$\Rightarrow f_{max} = 6\cdot\left(\frac{5+\sqrt{13}}{2}\right) = \underline{\underline{25.8}}$

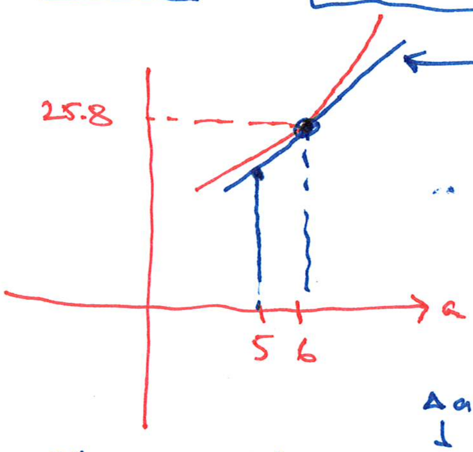
Envelope thm:

max $f(x,y,z) = x^2+2xy+4y^2+z^2$ when $x^2+y^2+z^2 = 6$

$\Rightarrow f_{max} = 3(5+\sqrt{13}) \simeq \underline{25.8}$ $\quad \lambda = \frac{5+\sqrt{13}}{2}$ $\leftarrow \underline{a=6}$

What if: $x^2+y^2+z^2 = 5$ $\quad$ " $f^*(6)$ $\quad$ " $\lambda^*(6)$ $\quad$ $f^*(a)$: max value fn.

Consider: $\boxed{\text{max } f(x,y,z) \quad \text{when } x^2+y^2+z^2 = a}$



slope $= \frac{df^*(a)}{da} = L'_a(\underline{x}^*(a);\lambda^*(a))$ Env. thm

$= \lambda^*(a)$

$= \frac{5+\sqrt{13}}{2} \simeq 4.3$

at $\underline{a=6}$

$$L = \underline{x}^T A \underline{x} - \lambda(x^2+y^2+z^2-a) = \ldots + \lambda a$$

$L'_a = \lambda$

$L'_a(\underline{x}^*(a);\lambda^*(a)) = \lambda^*(a)$

$$f^*(5) \simeq f^*(6) + \overset{\Delta a}{(-1)} \cdot \overset{\frac{df^*(a)}{da}}{\frac{5+\sqrt{13}}{2}}$$
$$\cong 25.8 + (-1)\cdot 4.3 \simeq \underline{\underline{21.5}}$$

2 Minimum variance portfolio problems

Ex: $\min f(x,y) = 0.07x^2 + 0.08xy + 0.06y^2$ when $x+y=1$

$$= (x\ y)\begin{pmatrix} 0.07 & 0.04 \\ 0.04 & 0.06 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{matrix} A & r_A \\ B & r_B \end{matrix}\right\} \quad \text{covariance matrix} \quad \Sigma = \begin{pmatrix} 0.07 & 0.04 \\ 0.04 & 0.06 \end{pmatrix}$$

$f = \underline{x}^T A \underline{x}$ $\qquad B\underline{x} = 1$ $\qquad \mathcal{L} = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 1)$

$A = \frac{1}{100}\begin{pmatrix} 7 & 4 \\ 4 & 6 \end{pmatrix}$ $\qquad B = (1 \quad 1)$

FOC: $2A\underline{x} - \lambda \cdot B^T = \underline{0}$

C: $B\underline{x} = 1$

$$\Rightarrow \quad 2A\underline{x} = \lambda B^T = \lambda \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\underline{x} = \frac{\lambda}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A^{-1}A\underline{x} = \frac{\lambda}{2} A^{-1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{x} = \frac{\lambda}{2} \frac{100}{26}\begin{pmatrix} 6 & -4 \\ -4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{\lambda}{\cancel{2}} \frac{\cancel{100}^{50}}{26}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{50\lambda}{26} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$A = \frac{1}{100}\begin{pmatrix} 7 & 4 \\ 4 & 6 \end{pmatrix}$ $\quad |A| = \frac{26}{100^2} \neq 0$

$$A^{-1} = \frac{100^2}{26} \cdot \frac{1}{\cancel{100}}\begin{pmatrix} 6 & -4 \\ -4 & 7 \end{pmatrix}$$

$$= \frac{100}{26}\begin{pmatrix} 6 & -4 \\ -4 & 7 \end{pmatrix}$$

$$\frac{50\lambda}{26} \cdot 2 + \frac{50\lambda}{26} \cdot 3 = 1$$

$$\frac{50\lambda}{26}(5) = 1 \qquad \frac{250\lambda}{26} = 1$$

$$\lambda = \frac{26}{250}$$

Cand. pt: $(2/5, 3/5; 26/250)$

$$\underline{x} = \frac{\cancel{50}}{\cancel{26}} \cdot \frac{\cancel{26}}{\cancel{250}\, 5} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

SOC: $h(x,y) = h(x,y;\, {}^{26}/_{250}) = \underline{x}^T A \underline{x} - \frac{26}{250}(x+y-1)$

$H(h) = 2A = 2 \cdot \frac{1}{100} \begin{pmatrix} 7 & 4 \\ 4 & 6 \end{pmatrix}$ pos. defn.

$\begin{pmatrix} 7 & 4 \\ 4 & 6 \end{pmatrix}$ $D_1 = 7$, $D_2 = 26$ pos. defn.

h convex $\underset{SOC}{\Rightarrow}$ $f_{min} = \underline{\underline{f(2/5, 3/5)}}$