| | Plan | |
|---|---|---|
| 1 | Key Problems: | 7.3b, 7.4c, 8.1, 8.2a, 9.1, 9.2b, 9.3c, 9.4 |
| 2 | [E]: | 6.3d, 6.4, 6.5 |
| 3 | Final exams: | 11/2017 Q3ab,4bc, 11/2018 Q3b,4b, 01/2019 Q5<br>11/2019 Q4ab, 01/2022 Q3, 04/2022 Q1c,3e |

## ① Key Problems

7.4 c) $x^3+y^3+z^3=0$ (C)

$0^3+0^3+0^3=0$ ok

NDCQ: $\text{rk}\begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \end{pmatrix}=1$

NDCQ fails: $\text{rk}=0$

$\Updownarrow$

$3x^2=3y^2=3z^2=0$

$(x,y,z)=\underline{(0,0,0)}$

$\Rightarrow$ One adm. pt where NDCQ fails: $\underline{(0,0,0)}$

8.1 a) max $f=x-2y+z$ when $x^2+y^2+z^2\le 3$ (std. form)

Compact $-\sqrt{3}\le x\le\sqrt{3}$, $y$, $z$

$\Rightarrow$ there is a max (EVT)

$\mathcal{L}=x-2y+z-\lambda(x^2+y^2+z^2-3)$

FOC: $\mathcal{L}'_x=1-\lambda\cdot 2x=0$
$\mathcal{L}'_y=-2-\lambda\cdot 2y=0$
$\mathcal{L}'_z=1-\lambda\cdot 2z=0$

C: $x^2+y^2+z^2\le 3$

CSC: $\lambda\ge 0$
$\lambda\cdot(x^2+y^2+z^2-3)=0$

B: $\left.\begin{array}{l} x^2+y^2+z^2=3 \\ \lambda\ge 0 \end{array}\right\}$ $x=\frac{1}{2\lambda}$ $z=\frac{1}{2\lambda}$, $y=\frac{-2}{2\lambda}$

$\left(\frac{1}{2\lambda}\right)^2+\left(\frac{-2}{2\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=3 \quad |\cdot(2\lambda)^2$

$1^2+(-2)^2+1^2=3\cdot(2\lambda)^2 \quad |:3$

$(2\lambda)^2=2 \quad 2\lambda=\pm\sqrt{2}=\sqrt{2}$

$\lambda=\sqrt{2}/2$

$x=\frac{1}{\sqrt{2}}$ $y=\frac{-2}{\sqrt{2}}$ $z=\frac{1}{\sqrt{2}}$ $\lambda=\sqrt{2}/2$

NB: $\left.\begin{array}{l} x^2+y^2+z^2<3 \\ \lambda=0 \end{array}\right\}$ $1-0=0$
no cand. pts

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$$f(1/\sqrt{2}, -2/\sqrt{2}, 1/\sqrt{2}) = \frac{1}{\sqrt{2}} + \frac{4}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 6/\sqrt{2} = 3 \cdot \frac{2}{\sqrt{2}} = 3\sqrt{2}$$

NDCQ: $\text{rk}\,(2x \;\; 2y \;\; 2z) = 1$

fails if $x = y = z = 0$ $\longrightarrow$ B: $x^2 + y^2 + z^2 = 3$

$0 = 3$ not possible

no condition

NB: $x^2 + y^2 + z^2 < 3$

$\Rightarrow f_{max} = 3\sqrt{2}$

8.1 b) max $f = xz + yw$ when $\begin{cases} x^2 + y^2 \le 1 \\ 4z^2 + 9w^2 \le 36 \end{cases}$ (std. form)

$$\mathcal{L} = xz + yw - \lambda_1(x^2 + y^2 - 1) - \lambda_2(4z^2 + 9w^2 - 36)$$

FOC:

$$\boxed{\begin{aligned} \mathcal{L}'_x &= z - \lambda_1 \cdot 2x = 0 \quad (i) \\ \mathcal{L}'_y &= w - \lambda_1 \cdot 2y = 0 \quad (ii) \\ \mathcal{L}'_z &= x - \lambda_2 \cdot 8z = 0 \quad (iii) \\ \mathcal{L}'_w &= y - \lambda_2 \cdot 18w = 0 \quad (iv) \end{aligned}}$$

C: $x^2 + y^2 \le 1$

$4z^2 + 9w^2 \le 36$

CSC: $\lambda_1 \ge 0, \; \lambda_2 \ge 0$

$\lambda_1(x^2 + y^2 - 1) = 0$

$\lambda_2(4z^2 + 9w^2 - 36) = 0$

(i)+(iii): $z = 2\lambda_1 x$

$x - 8\lambda_2(2\lambda_1 x) = 0$

$x \cdot (1 - 16\lambda_1\lambda_2) = 0$

a) $x = 0$, $z = 0$

or

b) $\lambda_1\lambda_2 = 1/16$

(ii)+(iv): $w = 2\lambda_1 y$

$y - 18\lambda_2(2\lambda_1 y) = 0$

$y \cdot (1 - 36\lambda_1\lambda_2) = 0$

c) $y = 0$, $w = 0$

or

d) $\lambda_1\lambda_2 = 1/36$

a) and c): $x = y = z = w = 0 \quad \lambda_1 = \lambda_2 = 0 \quad f = 0$

a) and d): $x = z = 0$, $\lambda_1\lambda_2 = 1/36$ $\rightarrow$ $y = \pm 1$, $w = \pm 2$ $\quad w, y$ same sign

$x^2 + y^2 = 1 \quad 4z^2 + 9w^2 = 36 \quad \lambda_1 = \frac{w}{2y} = 1 \quad \lambda_2 = 1/36$

$(0,1,0,2;1,1/36)$ $f=2$

$(0,-1,0,-2;1,1/36)$ $f=2$

b) and c): $y=w=0$, $\lambda_1\lambda_2=1/16$ $\Rightarrow$ $x=\pm 1, z=\pm 3$ same sign

$x^2+y^2=1$, $4z^2+9w^2=36$ $\quad \lambda_1 = \frac{z}{2x} = \frac{3}{2}$ $\quad \lambda_2 = \frac{1}{24}$

$(1,0,3,0;3/2,1/24)$ $f=3$ ← Best cand. pts.

$(-1,0,-3,0;3/2,1/24)$ $f=3$

b) and d): $\lambda_1\lambda_2 = 1/36 = 1/16$ impossible

test these cand. pts

SOC: $h(x,y,z,w) = L(x,y,z,w; 3/2, 1/24)$

$= xz + yw - \frac{3}{2}(x^2+y^2-1) - \frac{1}{24}(4z^2+9w^2-36)$

$$H(h) = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -3/4 \end{pmatrix}$$

$D_1 = -3$

$D_2 = 9$

$D_3 = -3(0) = 0$

$D_4 = 1 \cdot \begin{vmatrix} -3 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1/3 & 0 \end{vmatrix} - \frac{3}{4} \cdot 0 = 0$

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel$

$\quad \quad \quad \quad \quad \quad \quad -1 \cdot 0$

$\Delta_1 = -3, -3, -1/3, -3/4 \leq 0$

$\Delta_2 = 9, 1, 1/4, 0, 9/4-1, 9/4 \geq 0$

$\Delta_3 = 0, -1/3(9/4-1), -3(9/4-1), 0 \leq 0$

$\Delta_4 = 0 \geq 0$

$H(h)$ neg. semidefin. at all pts

$\Downarrow$

$h$ concave

$\Downarrow$ SOC

$f_{max} = \underline{\underline{3}}$

$$M_{124} = \begin{vmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3/4 \end{vmatrix} = -3 \cdot (9/4 - 1)$$

$$M_{134} = \begin{vmatrix} -3 & 1 & 0 \\ 1 & -1/3 & 0 \\ 0 & 0 & -3/4 \end{vmatrix} = -3/4(1-1) = 0$$

**9.1** max $f = 2x^2 - 4y^2 - 2z^2$ when $x^4 + y^4 + z^4 \le 16$

a) $L = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - 16)$

FOC: $L'_x = 4x - \lambda \cdot 4x^3 = 0$ C: $x^4 + y^4 + z^4 \le 16$

$L'_y = -8y - \lambda \cdot 4y^3 = 0$ CSC: $\lambda \ge 0$

$L'_z = -4z - \lambda \cdot 4z^3 = 0$ $\lambda(x^4 + y^4 + z^4 - 16) = 0$

$4x(1 - \lambda x^2) = 0$ $x = 0$ or $\lambda = 1/x^2$

$4y(-2 - \lambda y^2) = 0$ $y = 0$ or $\cancel{\lambda = -2/y^2}$

$4z(-1 - \lambda z^2) = 0$ $z = 0$ or $\cancel{\lambda = -1/z^2}$

i) $x = 0$: $(0,0,0)$ $\lambda = 0$ $f = 0$

ii) $\lambda = 1/x^2$: $\lambda > 0$ $x^4 + y^4 + z^4 = 16$ $x^4 = 16$ $x = \pm\sqrt[4]{16} = \pm 2$

$(x \neq 0)$ $y = 0$ $z = 0$

$\rightarrow (\pm 2, 0, 0 ; 1/4)$ $f = 8$ best cand. pts.

$f_{max} = 8$ at $(\pm 2, 0, 0)$ with $\lambda = 1/4$ since

i) D is compact $-2 \le x, y, z \le 2$ $\Rightarrow$ (EVT) there is a max.

ii) NDCQ holds at all adm. pts.

b) $x^4 + y^4 + z^4 \le 20$: $L = f - \lambda(x^4 + y^4 + z^4 - a)$

$f^*(a)$: $f^*(16) = 8$

$f^*(20) \cong 8 + 4 \cdot L'_a(\underline{x}^*(a); \lambda^*(a))$

$= 8 + 4 \cdot 1/4 = 9$

$f(x,y,z) = ax^2 - 4y^2 - 2z^2$ with $a = 1$

$L = ax^2 - 4y^2 - 2z^2 - \lambda \ldots$

$L'_a = x^2$

$f^*(2) = 8$

$f^*(1) \simeq 8 + (-1) \cdot L'_a(\underline{x}^*(a); \lambda^*(a))$

$= 8 + (-1) \cdot 4 = 4$

9.2 b) $f(\underline{x}) = \underline{x}^T A \underline{x} + 1$

$f_{min} = f(0,0,0,0) = \underline{1}$

no global max

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 6 & 0 \\ -1 & 1 & 0 & 3 \end{pmatrix} \qquad \begin{aligned} D_1 &= 3 \\ D_2 &= 2 \\ D_3 &= 0 \\ D_4 &= 0 \end{aligned}$$

RRC:

A pos semidef.

$$|A| = 1 \cdot \underbrace{\begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ 2 & 6 & 0 \end{vmatrix}}_{= 0} + 1 \cdot \underbrace{\begin{vmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \\ 4 & 6 & 0 \end{vmatrix}}_{= 0} + 3 \cdot \underbrace{D_3}_{= 0} = 0$$

$$\begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 6 & 0 \\ -1 & 1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & -2 & -2 & -4 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \underline{\operatorname{rk} A = 2}$$

$$V_f = \underline{\underline{[1, \to)}}$$

9.3 $f = -4x^2 - 10y^2 - 5z^2 - 5w^2 + 4xz + 4xw - 4yz + 4yw + 6zw$ when $x^2+y^2+z^2+w^2 = 6$

c) max $f(x,y,z,w)$ when $x^2+y^2+z^2+w^2 = 6$

$$\underline{x}^T A \underline{x} \qquad \underline{x}^T I \underline{x} = 6$$

$$A = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T I \underline{x} - 6)$$

FOC: $L'(\underline{x}) = 2A\underline{x} - \lambda(2I\underline{x}) = 0 \quad |:2$ C: $\underline{x}^T I \underline{x} = 6$

$$A\underline{x} - \lambda I \underline{x} = \underline{0}$$

$$(A - \lambda I)\underline{x} = \underline{0}$$

$\underline{x} = \underline{0}$ or $|A - \lambda I| = 0$

$(x,y,z,w) = (0,0,0,0)$ not adm.

Find eigenvalues of $A$:

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq 0$

$|A| = 0 \Rightarrow$ at least one eigenvalue is $\lambda = 0$

$\Rightarrow$ max. eigenvalue is $\underline{\lambda = 0}$

$f_{max} = 6 \cdot 0 = \underline{\underline{0}}$

$\Uparrow$

i) there is a max by EVT

ii) the max pt. is an ordinary cand. pt.

iii) $\underline{x}$ is an eigenvect with eigv. $\lambda$:

$f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T \lambda \underline{x} = \lambda \cdot \underline{x}^T \underline{x} = \underline{\lambda \cdot 6}$

a) Is $f$ convex / concave:

$D_1 = -4 < 0$

$D_2 = 40 > 0$

$D_3 = \begin{vmatrix} -4 & 0 & 2 \\ 0 & -10 & -2 \\ 2 & -2 & -5 \end{vmatrix} = -4 \cdot (50 - 4) + 2\,(20) = -4 \cdot 46 + 40 < 0$

$D_4 = |A| = 0 \Rightarrow \overset{\text{RRC}}{\text{rk}\,A = 3} \Rightarrow A$ neg. semidefin. $f$ concave

**9.4.** $g(\underline{x}) = \underline{x}^T A \underline{x}$ $\qquad A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$

a) $D_1 = \underline{3}$

$D_2 = \underline{2}$

$D_3 = 3\cdot(7-4) - 1\cdot(7-8) + 4\cdot(2-4)$

$= 9 + 1 - 8 = \underline{2}$

$D_4 = 1\cdot\begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ 2 & 7 & 0 \end{vmatrix} + 1\cdot\begin{vmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \\ 4 & 7 & 0 \end{vmatrix} + 4\cdot 2$

$= 2\cdot 6 - 7\cdot 2 + 4\cdot 6 - 7\cdot 4 + 4\cdot 2 = -2 - 4 + 8 = \underline{2}$

$g$ is <u>pos. defn.</u>

b) $\mathcal{L} = x + y + z + w - \lambda(\underline{x}^T A \underline{x} - 18)$ $\qquad \underline{B = (1\ 1\ 1\ 1)}$

$= (1\ 1\ 1\ 1)\cdot\underline{x} - \lambda(\underline{x}^T A \underline{x} - 18) = B\underline{x} - \lambda(\underline{x}^T A \underline{x} - 18)$

<u>FOC:</u> $\mathcal{L}'(\underline{x}) = B^T - \lambda\cdot(2A\underline{x}) = \underline{0}$ $\qquad \boxed{B^T - 2\lambda\cdot A\underline{x} = \underline{0}}$

<u>C:</u> $g(\underline{x}) = \boxed{\underline{x}^T A \underline{x} \le 18}$

<u>CSC:</u> $\boxed{\lambda \ge 0, \quad \lambda\cdot(\underline{x}^T A \underline{x} - 18) = 0}$

c) <u>NDCQ:</u> $\operatorname{rk}(g'_x\ g'_y\ g'_z\ g'_w) = 1$ in the binding case $(\underline{x}^T A \underline{x} = 18)$

no condition in the non-binding case $(\underline{x}^T A \underline{x} < 18)$

<u>NDCQ fails:</u> $\underline{x}^T A \underline{x} = 18$ and $(g'_x\ g'_y\ g'_z\ g'_w) = \underline{0}$

$\underset{0}{g'_x} \ \underset{0}{g'_y} \ \underset{0}{g'_z} \ \underset{0}{g'_w}$

$2A\underline{x} = \underline{0} \quad |:2$

$A\underline{x} = \underline{0} \quad |A^{-1}$

$\underline{x} = \underline{0}$

Stat. pts for $g$: $g'(\underline{x}) = \underline{0}$

$2A\underline{x} = \underline{0}$

does not satisfy $\underline{x}^T A \underline{x} = 18$

<u>no pts</u>

d) KT cond.: $B^T - 2\lambda A\underline{x} = \underline{0}$

$\underline{x}^T A \underline{x} \le 18$

$\lambda \ge 0 \quad , \quad \lambda(\underline{x}^T A \underline{x} - 18) = 0$

$\lambda \neq 0$

$\underline{x}^T A \underline{x} = 18$: $\lambda \ge 0$ $\quad 2\lambda \cdot A\underline{x} = B^T \;\Big|\; : \frac{1}{2\lambda}$

$A\underline{x} = \frac{1}{2\lambda} B^T = \begin{pmatrix} \frac{1}{2\lambda} \\ \vdots \\ 1/2\lambda \end{pmatrix}$

$\underline{x}^T A \underline{x} < 18$: $\lambda = 0 \quad B^T = \underline{0} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ not possible

$$A\underline{x} = \begin{pmatrix} 1/2\lambda \\ 1/2\lambda \\ 1/2\lambda \\ 1/2\lambda \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 1 & 4 & -1 & t \\ 1 & 1 & 2 & 1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & t \\ 3 & 1 & 4 & -1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \begin{matrix} -3 \\ -4 \\ 1 \end{matrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & t \\ 0 & -2 & -2 & -4 & -2t \\ 0 & -2 & -1 & -4 & -3t \\ 0 & 2 & 2 & 5 & 2t \end{array}\right) \begin{matrix} -1 \\ -1 \\ 1 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & t \\ 0 & -2 & -2 & -4 & -2t \\ 0 & 0 & 1 & 0 & -t \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$\underline{w = 0}$ $\quad -2y - 2(-t) - 4\cdot 0 = -2t \quad x + 2t + 2(-t) + 1\cdot 0 = t$

$\underline{z = -t}$ $\quad -2y = -4t \quad \underline{x = t}$

$\underline{y = 2t}$

$\Rightarrow (x,y,z,w) = \underline{(t, 2t, -t, 0)}$ $\quad g(t, 2t, -t, 0) = 3t^2 + 4t^2 + 7t^2 + 4t^2 - 8t^2 + 8t^2 = \underline{2t^2} = 18$

Cand pt: ~~$(1/2, 1, -1/2, 0 ; 1/2)$~~ ~~$f = 0$~~ $\quad t^2 = 9 \quad t = \pm 3$

$(3, 6, -3, 0 ; 1/6)$ $\quad \underline{f = 6}$ $\quad \frac{1}{2\lambda} = \pm 3 = 3 \quad \underline{\lambda = 1/6}$

SOC: $h = Bx - \frac{1}{6}(\underline{x}^T A \underline{x} - 18)$

$H(h) = -\frac{1}{6}\cdot(2A) = -\frac{1}{3}A$ neg. defn since A is pos. defn from (a)

$\Downarrow$

$h$ concave

$\Downarrow$ SOC

$f_{max} = \underline{\underline{6}}$

AU: Is $\underline{x}^T A \underline{x} \le 18$ bounded? More difficult to check in this case.