	Plan	
1	Key Problems:	10.2c, 10.3b, 10.4c, 11.2d, 11.3, 12.1a, 12.3
2	Final exams:	04/2022 2ad, 01/2022 1,2c, 01/2020 3a, 12/2015 2c

1 Key Problems

10.2 c) $y' = 5y(1 - y/10) = \underbrace{5}_{f(t)} \cdot \underbrace{y \cdot \frac{10-y}{10}}_{g(y)}$ Seperable

$$\frac{10}{y(10-y)} y' = 5$$

$$\int \frac{10}{y(10-y)} dy = \int 5 \, dt$$

$$\frac{10}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y} \quad | \cdot y(10-y)$$

$$10 = A(10-y) + By$$

$$10 = \underbrace{10A}_{=10} + \underbrace{(B-A)y}_{=0}$$

$$A = 1 \qquad B = 1$$

$$\int \frac{1}{y} + \frac{1}{10-y} \, dy = 5t + C$$

$$\ln|y| - \ln|10-y| = \ln\left|\frac{y}{10-y}\right| = 5t + C \quad | \exp(\cdot)$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$
$$\ln a + \ln b = \ln(a \cdot b)$$
$$n \cdot \ln(a) = \ln(a^n)$$

$$\left|\frac{y}{10-y}\right| = e^{5t+C} = e^{5t} \cdot e^{C}$$

$$\frac{y}{10-y} = \pm e^{C} e^{5t} = K \cdot e^{5t} \quad | \cdot (10-y)$$

$$y = (10-y) \cdot Ke^{5t} = 10 \cdot Ke^{5t} - y \cdot Ke^{5t}$$

$$\frac{y\cancel{(1+Ke^{5t})}}{\cancel{1+Ke^{5t}}} = \frac{10 \cdot Ke^{5t}}{1+Ke^{5t}} \qquad y = 10 \cdot \frac{Ke^{5t}}{1+Ke^{5t}}$$

10.3 b

$$y' - 2ty = 4t \quad | \cdot u$$

$$(y \cdot e^{-t^2})' = 4te^{-t^2}$$

$$y \cdot e^{-t^2} = \int 4t e^{-t^2} dt$$

$$\boxed{\begin{matrix} u = -t^2 \\ du = -2t\,dt \end{matrix}}$$

$$= \int 4\cancel{t} e^u \frac{dt}{-2\cancel{t}}$$

$$= \int -2e^u du$$

linear

i) integrating factor:

$$\int -2t\,dt = -t^2 + C$$

$u = e^{-t^2}$ int. factor

ii) superposition principle: works for a linear diff. equs

$$y = y_h + y_p$$

but characteristic equ. only works when $a(t) = a$ is constant

$$y \cdot e^{-t^2} = -2e^u + C = -2e^{-t^2} + C$$

$$y = e^{t^2}(-2e^{-t^2} + C) = \underline{\underline{-2 + C \cdot e^{t^2}}}$$

10.4 c

$$\underbrace{\frac{y(1-2\ln t)}{t^3}}_{= h'_t} + \underbrace{\frac{\ln t}{t^2}}_{= h'_y} \cdot y' = 0 \qquad p + q \cdot y' = 0$$

$$\boxed{\begin{matrix} (1)\ h'_t = \dfrac{y(1-2\ln t)}{t^3} \\ (2)\ h'_y = \dfrac{\ln t}{t^2} \end{matrix}} \Rightarrow h = \frac{\ln t}{t^2} \cdot y + a(t)$$

$$\Rightarrow h'_t = \left(\frac{\ln t}{t^2}\right)'_t \cdot y + a'(t) = y \cdot \frac{(1-2\ln t)}{t^3} \quad \text{ok if } a(t) = 0$$

$$\left(\frac{\ln t}{t^2}\right)' = \frac{\frac{1}{t} \cdot t^2 - \ln t \cdot 2t}{t^4} = \frac{t - 2t\ln t}{t^4} = \frac{\cancel{t} \cdot (1-2\ln t)}{t^{\cancel{4}3}}$$

$$= \frac{1-2\ln t}{t^3}$$

Conclusion: Exact $\quad h = \dfrac{\ln t}{t^2} y = C \Rightarrow y = \underline{\underline{C \cdot \dfrac{t^2}{\ln t}}}$

11.2 d) $y'' - y = t^2$ second order linear

$y = y_h + y_p = \underline{C_1 e^t + C_2 e^{-t} - t^2 - 2}$ Superposition principle

y_h: $y'' - y = 0$

$r^2 - 1 = 0 \quad r = \pm 1 \quad \Rightarrow y_h = \underline{C_1 \cdot e^t + C_2 e^{-t}}$

y_p: $y'' - y = t^2$

$h(t) = t^2 \dashrightarrow$ $y = At^2 + Bt + C$

$h'(t) = 2t$ $\quad y' = A \cdot 2t + B$

$h''(t) = 2$ $\quad y'' = 2A$

$2A - (At^2 + Bt + C) = t^2$

$\underbrace{-At^2}_{=1} \underbrace{-Bt}_{=0} + \underbrace{(2A - C)}_{=0} = t^2$

$-A = 1 \Rightarrow \underline{A = -1}$

$-B = 0 \quad \underline{B = 0}$

$2A - C = 0 \quad C = 2A = 2(-1) = \underline{-2}$

$\rightarrow y_p = \underline{-t^2 - 2}$

11.3 a) $y = \sqrt{t^2 - 3} = \sqrt{u} = u^{1/2}, \quad u = t^2 - 3$

$y' = \frac{1}{2} u^{-1/2} \cdot u' = \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}t = \frac{t}{\sqrt{t^2 - 3}}$

$y' = \frac{t}{\sqrt{t^2 - 3}} = \frac{t}{y}$

$$\underline{\underline{y' = \frac{t}{y}}}$$

b) $y = \frac{2}{1 - t^2} \Rightarrow y' = 2 \cdot (-1)(1 - t^2)^{-2} \cdot (-2t) = \frac{4t}{(1 - t^2)^2} = t y^2$

$$\underline{\underline{y' = t y^2}}$$

12.1 a)
$$y_1' = 2y_1 + 5y_2$$
$$y_2' = 5y_1 + 2y_2$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$:
$$\lambda^2 - 4\lambda - 21 = 0$$
$$(\lambda - 7)(\lambda + 3) = 0$$
$$\lambda_1 = 7, \quad \lambda_2 = -3$$

Bases of E_{λ_i}:

$\lambda_1 = 7$: $\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \longrightarrow \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$: $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \longrightarrow \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

General solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{7t} + C_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$

$$y_1 = C_1 e^{7t} - C_2 e^{-3t}$$
$$y_2 = C_1 e^{7t} + C_2 e^{-3t}$$

12.3. $y''' + 4y'' + y' - 6y = 0$

$r^3 + 4r^2 + r - 6 = 0$

$$\left.\begin{array}{l} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{array}\right\} \begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 6y_1 - y_2 - 4y_3 \end{array}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -1 & -4-\lambda \end{vmatrix} = 0$$

$\cancel{-\lambda^3 + 4\lambda^2 - \lambda - 6 = 0}$

$r = 1$ is a solution $\rightarrow$

$$r^3 + 4r^2 + r - 6 : r - 1 = r^2 + 5r + 6$$
$$\underline{r^3 - r^2}$$
$$5r^2 + r - 6$$
$$\underline{5r^2 - 5r}$$
$$6r - 6$$
$$\underline{6r - 6}$$
$$0$$

$$(r-1)(r^2 + 5r + 6) = 0$$
$$r = 1 \quad r = -2 \quad r = -3$$

$$\Rightarrow y = C_1 e^{t} + C_2 \cdot e^{-2t} + C_3 e^{-3t}$$

Using systems:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix}}_{= A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \text{where} \quad \begin{matrix} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{matrix}$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = -2, \lambda_3 = -3$

Eigenvectors:

$\lambda = 1$: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -1 & -5 \end{pmatrix} \to \underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = -2$: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 6 & -1 & -2 \end{pmatrix} \to \underline{v_2} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ $\qquad 2x = -y$, $2y = -z$

$\lambda = -3$: $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 6 & -1 & -1 \end{pmatrix} \to \underline{v_3} = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$ $\qquad 3x = -y$, $3y = -z$

General solution:

$$\begin{matrix} y \to \\ y' \to \\ y'' \to \end{matrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \cdot \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2t} + c_3 \cdot \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix} e^{-3t}$$

$$\Downarrow$$

$$y = c_1 \cdot e^{t} + c_2 e^{-2t} + c_3 \cdot e^{-3t}$$

② Final exams

04/2022, 2

a) $y'' - 3y' + 2y = 6e^{-t}$

$y = y_h + y_p = \underline{C_1 e^{t} + C_2 e^{2t} + e^{-t}}$

y_h: $r^2 - 3r + 2 = 0 \Rightarrow y_h = \underline{C_1 e^{t} + C_2 e^{2t}}$

$r = 1, \ r = 2$

y_p: $y'' - 3y' + 2y = 6e^{-t}$

$(Ae^{-t}) - 3(-Ae^{-t}) + 2(Ae^{-t}) = 6e^{-t}$

$h = 6e^{-t}$
$h' = -6e^{-t}$
$h'' = 6e^{-t}$

$$\boxed{\begin{aligned} y &= A\cdot e^{-t} \\ y' &= -Ae^{-t} \\ y'' &= Ae^{-t} \end{aligned}}$$

$\underbrace{(A + 3A + 2A)}\cdot e^{-t} = 6e^{-t}$

$6A = 6$
$A = 1$ $\Rightarrow y_p = \underline{e^{-t}}$

d) $y' = \begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} y + \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$ (lin. sys. of diff. equ. inhomogeneous)

Method:

i) Eq. state: $\begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} y + \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x = -9$
$y = -2$
$z = 1$

Gauss: $\left(\begin{array}{ccc|c} 2 & -4 & -11 & 3 \\ -2 & 3 & 10 & -2 \\ 1 & -4 & -10 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -4 & -10 & 1 \\ -2 & 3 & 10 & -2 \\ 2 & -4 & -11 & 3 \end{array}\right) \begin{matrix} 2 \\ -2 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|c} 1 & -4 & -10 & 1 \\ 0 & -5 & -10 & 0 \\ 0 & 4 & 9 & 1 \end{array}\right) : (-5) \rightarrow \left(\begin{array}{ccc|c} 1 & -4 & -10 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 9 & 1 \end{array}\right) -4 \rightarrow \left(\begin{array}{ccc|c} 1 & -4 & -10 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$

Eq. state: $y_e = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

Stability: A has eigenvalues $\lambda = 1$, $\lambda = -5$, $\lambda = -1$ from Q1.

$\Rightarrow$ y_e is unstable since $\lambda = 1 > 0$

Q1 (2020, 3a):

$$y'' - 11y' + 18y = 9t^2 - 11t + 10$$

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{9t} + \frac{1}{2}t^2 + \frac{1}{2}$$

y_h: $y'' - 11y' + 18y = 0$

$$r^2 - 11r + 18 = 0$$
$$(r-2)(r-9) = 0$$
$$r = 2, \quad r = 9 \quad \longrightarrow \quad y_h = C_1 e^{2t} + C_2 e^{9t}$$

y_p: $y'' - 11y' + 18y = 9t^2 - 11t + 10$ $\leftarrow$ $\begin{cases} y = At^2 + Bt + C \\ y' = 2At + B \\ y'' = 2A \end{cases}$

$$2A - 11(2At + B) + 18(At^2 + Bt + C) = 9t^2 - 11t + 10$$

$$\underbrace{18A}_{=9} \cdot t^2 + \underbrace{(18B - 22A)}_{=-11} t + \underbrace{(2A - 11B + 18C)}_{=10} = 9t^2 - 11t + 10$$

$A = 1/2$

$18B = -11 + 22A = 0$, $B = 0$

$1 - 0 + 18C = 10$, $18C = 9$, $C = 1/2$

$$y_p = \frac{1}{2}t^2 + \frac{1}{2}$$

12/2015, 2c)

$$(4yt + 4t^3 + 2t) + (2y - 1 + 2t^2)\,y' = 0 \quad , \quad y(1) = 0$$

$$p + q \cdot y' = 0$$

initial cond.

$$\boxed{\begin{aligned} (1)\quad & h'_t = 4yt + 4t^3 + 2t \\ (2)\quad & h'_y = 2y - 1 + 2t^2 \end{aligned}}$$

$$(1)\quad h = y \cdot 2t^2 + t^4 + t^2 + \alpha(y)$$

$$(2)\quad h'_y = 2t^2 + 0 + 0 + \alpha'(y) = 2y - 1 + 2t^2$$

$$\alpha'(y) = 2y - 1$$

$$\alpha(y) = y^2 - y$$

$\Downarrow$

Exact with $h = y \cdot 2t^2 + t^4 + t^2 + y^2 - y = C$

$$y^2 + (2t^2 - 1)y + (t^4 + t^2) = C$$

$\cancel{y}(1) = 0$:

$t = 1,\ y = 0$

$$0^2 + (2-1)\cdot 0 + (1+1) = C$$

$$C = 2$$

$\Downarrow$

$$y^2 + (2t^2 - 1)y + (t^4 + t^2 - 2) = 0$$

$$Ay^2 + By + C = 0$$

$$y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$y = \frac{-(2t^2 - 1) \pm \sqrt{(2t^2-1)^2 - 4(t^4 + t^2 - 2)}}{2 \cdot 1}$$

$$= \frac{1 - 2t^2}{2} \pm \frac{1}{2}\sqrt{\cancel{4t^4} - \cancel{4t^2} + 1 - \cancel{4t^4} - 4t^2 + 8}$$

$$y = \frac{1 - 2t^2}{2} \pm \frac{1}{2}\sqrt{9 - 8t^2}$$

$t = 1$: $-\frac{1}{2} \pm \frac{1}{2} \cdot 1 = -\frac{1}{2} + \frac{1}{2} = 0$

$$y = \frac{1 - 2t^2}{2} + \frac{1}{2}\sqrt{9 - 8t^2}$$

01/2022, Q2c)

$$(y^2-3t^2y)+(2ty-t^3)y'=0$$

$$\boxed{\begin{aligned} h'_t &= y^2-3t^2y \\ h'_y &= 2ty-t^3 \end{aligned}}$$

$$h = y^2t - t^3y + \varphi(y)$$

$$h'_y = \underline{2yt} \; \underline{-t^3} + \varphi'(y) = \underline{2ty} \; \underline{-t^3}$$

$$\Downarrow$$

$$\underline{h = y^2t - t^3y = C}$$

$$ty^2 - t^3y - C = 0$$

$$\Downarrow$$

$$y = \frac{t^3 \pm \sqrt{t^6 - 4t(-C)}}{2t}$$

$$= \frac{t^2}{2} \pm \frac{\sqrt{t^6+4Ct}}{2t}$$