
 Plan

- 1 Linear systems and their geometry
 - 2 Gaussian elimination
 - 3 Rank of a matrix
 - 4 Homogeneous linear systems
-

① Linear systems and their geometry

Ex:

$$\begin{aligned} x + y + z + w &= 5 \\ 2x - 3y + \cancel{z} - w &= 0 \\ x + 6y + 2z + 4w &= h \end{aligned}$$

3x4 linear system
with parameter h
and variables x, y, z, w

Defn: A linear equation in n variables (say $x_1, x_2, \dots, x_n$) is an eqn. of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b}$$

where $a_1, a_2, \dots, a_n, b$ are given numbers. It is called degenerated if $a_1 = a_2 = \dots = a_n = 0$.

Defn: The solutions of this linear eqn. are the n -tuples $(x_1, x_2, \dots, x_n)$ that satisfy the eqn.

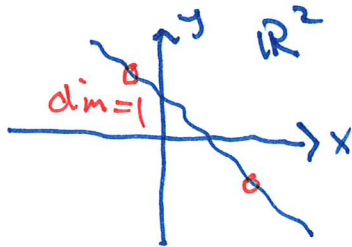
Assuming the eqn. is not degenerated

{	If $n=2$, then the solutions form a <u>line</u>
	If $n=3$, — — a <u>plane</u>
	If $n>3$, — — a <u>hyperplane</u>

$n=2$: $ax+by=c$

$$by=c-ax$$

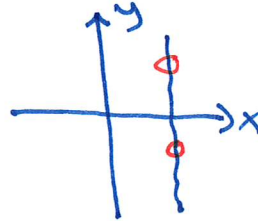
$b \neq 0$: $y = \frac{c}{b} - \frac{a}{b}x$



$b=0, a \neq 0$

$$ax=c$$

$$x=c/a$$



$b=0, a=0$

degenerated case

$$0 \cdot x + 0 \cdot y = c$$

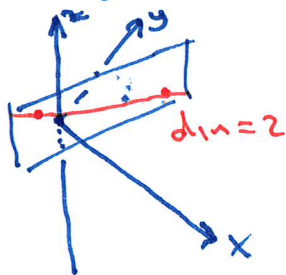
$c \neq 0$

no
Solutions

$c=0$

all of
 $\mathbb{R}^2$ are
Soln's.

$n=3$: $ax+by+cz=d$



Result: Linear property: with $P \neq Q$

If P and Q are solutions of a linear equation, then the line $[P, Q]$ that goes thr. P and Q consists of solutions of the same eqn.

Defn An $m \times n$ linear system is a system of m linear equations in n variables (say $x_1, x_2, \dots, x_n$). It can be written in std form as

$$(*) \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij}, b_i are given numbers.

Defn A solution of the linear system $(*)$ is an n -tuple $(x_1, \dots, x_n)$ that are solutions of each linear eqn. in $(*)$ simultaneously.

The linear system (*) is called consistent if it has solutions.

Theorem: Any linear system satisfy the linear property. In particular, any linear system has either

- | | |
|--------------------------------|----------------|
| i) no solutions | } inconsistent |
| ii) one unique solution | |
| iii) infinitely many solutions | } consistent |

② Gaussian elimination

general method for solving linear systems

Representations of linear systems:

$m \times n$ linear system (*) :

$(A | \underline{b})$
augmented matrix of (*)

$A \cdot \underline{x} = \underline{b}$
matrix form of (*)

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$m \times n$ coeff. matrix of (*)

Ex.

$$\begin{aligned} x + y + z + w &= 5 \\ 2x - 3y + z - w &= 0 \\ x + 6y + 2z + 4w &= h \end{aligned}$$

3x4 lin. sys.

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -3 & 1 & -1 & 0 \\ 1 & 6 & 2 & 4 & h \end{array} \right)$$

augmented matrix

Method: Gaussian elimination

- i) Write down the augmented matrix of the linear system
- ii) Elementary row operations until we obtain an echelon form.
- iii) Solve the resulting linear system using back substitution

Ex: $x+y+z+w=5$
 $2x-3y+z-w=0$
 $x+6y+2z+4w=h$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -3 & 1 & -1 & 0 \\ 1 & 6 & 2 & 4 & h \end{array} \right) \begin{array}{l} \downarrow -2 \\ \downarrow -1 \end{array}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 5 & 1 & 3 & h-5 \end{array} \right) \begin{array}{l} \downarrow \\ \downarrow \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & h-15 \end{array} \right)$$

$h=15$

$$\begin{array}{cccc} x & y & z & w \end{array} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

echelon form

$$\begin{array}{rcl} x+y+z+w & = & 5 \\ -5y-z-3w & = & -10 \end{array}$$

x, y : basic
 z, w : free

$h \neq 15$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & h-15 \end{array} \right)$$

echelon form

$$\begin{array}{rcl} x+y+z+w & = & 5 \\ -5y-z-3w & = & -10 \\ 0 & = & h-15 \end{array}$$

No solutions

Defn:

An elementary row operation is

either:

- i) switch two rows
- ii) multiply one row with $c \neq 0$
- iii) add a multiple of one row to another row

Defn:

A matrix is in echelon form if

- a) all zero rows are in the bottom of the matrix
- b) each pivot (= first non-zero element of a row) are further to the right than the pivots in the rows above

$$\underline{-5y - z - 3w = -10}$$

$$\underline{-5y = z + 3w - 10}$$

$$\underline{-5 \quad \quad -5}$$

$$y = \underline{-\frac{1}{5}z - \frac{3}{5}w + 2}$$

$$\underline{x + y + z + w = 5}$$

$$x = -y - z - w + 5$$

$$= \frac{1}{5}z + \frac{3}{5}w - 2 - z - w + 5$$

$$x = \underline{-\frac{4}{5}z - \frac{2}{5}w + 3}$$

Conclusion:

$n \neq 15$: no solutions

$n = 15$: $(x, y, z, w) =$

$$\left(-\frac{4}{5}z - \frac{2}{5}w - 3, -\frac{1}{5}z - \frac{3}{5}w + 2, z, w\right)$$

where z, w are free
(infinitely many solutions)

2-dim = plane

Summary:

Defn: Pivot positions = the ~~first~~ positions in the matrix that have pivots in the echelon form

Result:

- i) The linear system is inconsistent (no solutions) if and only if there is a pivot position in the last column.
- ii) If the linear system is consistent, then there ~~are~~ one unique solution if there is a pivot position in each variable col.

b) Infinitely many solutions if at least one variable col. lacks a pivot position.

In case b): A variable is called free if there is no pivot position in the corr. column, and basic otherwise.

Defn: In case b), the number of free variables is called the number of degrees of freedom,
= dimension of the set of solutions

③ Rank of a matrix

Note: — any matrix can be transformed into an echelon form using elementary row operations
— the echelon form obtained is not unique, but the pivot positions are unique

Defn: The rank of a matrix is the number of pivot positions in an echelon form.

Ex:
$$A = \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} \swarrow -1 \\ \swarrow -1 \end{matrix} \rightarrow \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 0 & \textcircled{1} & 3 \\ 0 & 2 & 8 \end{pmatrix} \begin{matrix} \swarrow -1 \\ \swarrow -2 \end{matrix} \rightarrow \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 0 & \textcircled{1} & 3 \\ 0 & 0 & \textcircled{2} \end{pmatrix}$$

echelon form

$\text{rk } A = 3$

 The rank of A

Ex:

$$A = \begin{pmatrix} 4 & 1 & 3 \\ 2 & -1 & 7 \\ 3 & 1 & 5 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 0 & -2 \\ 2 & -1 & 7 \\ 3 & 1 & 5 \end{pmatrix} \xrightarrow{\begin{matrix} -2 \\ -3 \end{matrix}}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 11 \\ 0 & 1 & 11 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 11 \\ 0 & 0 & 22 \end{pmatrix} \quad \text{rk } A = \underline{\underline{3}}$$

Theorem

Let $A\underline{x} = \underline{b}$ be an $m \times n$ linear system in matrix form.

Then we have:

- i) The linear system is inconsistent $\Leftrightarrow \text{rk } A \neq \text{rk}(A|\underline{b})$
- ii) $\text{---} || \text{---}$ is consistent $\Leftrightarrow \text{rk } A = \text{rk}(A|\underline{b})$

and furthermore,

- * it has one unique solution $\Leftrightarrow \text{rk } A = n$
- * it has infinitely many solutions $\Leftrightarrow \text{rk } A < n$,
and in that case degrees of freedom = $\underline{n - \text{rk}(A)}$

In particular, the pivot positions determine how many solutions there are.

④ Homogeneous linear systems

Defn: A linear system $A\underline{x} = \underline{b}$ is homogeneous if $\underline{b} = \underline{0}$; that is, it can be written

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

Note: - Homogeneous lin. systems are ~~never~~ consistent,
It has the trivial solution $x_1 = x_2 = \dots = x_n = 0$.
- The number of free variables is $n - \text{rk } A$
has one unique soln.

Result: if $\text{rk } A = n$, then $A\underline{x} = 0 \forall$, and there is

one unique solution $\underline{x} = \underline{0}$

rk $A < n$, then $A\underline{x} = \underline{0}$ has $n - \text{rk } A$ free variables, and inf. many solutions

Defn. $\text{Null}(A) =$ the solutions of $A \cdot \underline{x} = \underline{0}$.
the nullspace
of A

Ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 \\ 3 & -3 & 6 & 1 \end{pmatrix}$

$\text{Null}(A) =$ all solutions of
 $A\mathbf{x} = \mathbf{0}$

↑
use Gauss to solve the
homogeneous lin. sys.

$$A = \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 \\ 3 & -3 & 6 & 1 & 0 \end{array} \right) \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}} \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & -6 & 3 & -2 & 0 \end{array} \right) \xrightarrow{R_3 - 2R_2}$$

$$\rightarrow \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 0 & \textcircled{1} & 2 & 0 \end{array} \right)$$

$\text{rk } A = 3 \quad n = 4 \quad n - \text{rk } A = 4 - 3 = 1$
(one degree of freedom)

$$\begin{aligned} x + y + z + w &= 0 \\ -3y + z - 2w &= 0 \\ z + 2w &= 0 \end{aligned}$$

$$\begin{aligned} x &= -y - z - w = \frac{4}{3}w + 2w - w = \frac{7}{3}w \\ -3y &= -z + 2w = 2w + 2w = 4w \quad y = -\frac{4}{3}w \\ z &= -2w \end{aligned}$$

not necessary
to include the
b-col. when
 $\mathbf{b} = \mathbf{0}$

Solutions = Null(A):

$(x, y, z, w) =$

$\left(\frac{7}{3}w, -\frac{4}{3}w, -2w, w \right)$

where w is free

Key Problems

You should do the Key Problems as soon as possible after each lecture. They test your understanding of the most important theories and methods presented in the lecture. You can work on Key Problems and get help in the Exercise session/TA session the following Monday (Campus Bergen: Wednesday).

Problem 1.

Solve the linear system $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination:

$$\text{a) } A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 1 & 8 \\ 4 & 5 & 9 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 11 \\ 15 \\ 23 \end{pmatrix}$$

$$\text{b) } A = \begin{pmatrix} 4 & 5 & 3 & 11 \\ 2 & 5 & 0 & 3 \\ 3 & 2 & 3 & 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 23 \\ 17 \\ 14 \end{pmatrix}$$

$$\text{c) } A = \begin{pmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 3 & 1 & 5 & 18 \\ 2 & 4 & 2 & 9 & 31 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 8 \\ 28 \\ 48 \end{pmatrix}$$

$$\text{d) } A = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 4 & 1 \\ 7 & 7 & 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 13 \\ 11 \\ 17 \end{pmatrix}$$

Problem 2.

Find the rank of the following matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 3 & 0 & 2 & 3 & 1 \\ 3 & 6 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \end{pmatrix}$$

Problem 3.

Let V be the set of solutions of the linear system $A\mathbf{x} = \mathbf{b}$ from Problem 1(b). What is the dimension of V ? Describe the set V geometrically.

Problem 4.

We consider the following linear system. Find all solutions that satisfies $x + w = y + z$:

$$\begin{array}{rrrrrr} x & + & y & + & 2z & + & 4w & = & 6 \\ x & + & 2y & + & 4z & - & 2w & = & 9 \\ x & + & 3y & + & 9z & + & 7w & = & 24 \end{array}$$

Problem 5.

We consider a 4×5 homogeneous linear system $A\mathbf{x} = \mathbf{0}$, and $\text{rk } A = 4$. How many solutions are there?

Exercise Problems

In addition to the Key Problems, you should try to do as many of the Exercise Problems as you can. They are usually problems from the textbook or exam problems. You can work with Exercise Problems and get help in the Exercise sessions/TA sessions on Mondays (Wednesdays). If you do not have time for all Exercise Problems in the following week, make sure to do them later in the semester as revision.

Problems from the textbook: [E] 1.1 - 1.17