Plan

1. Introduction to first order differential equations
2. Separable differential equations
3. Linear first order differential equations
4. Exact differential equations

Plenary Session 3:

Mon at 16-19

TA session:

16-18 in D1-065

① Introduction to differential equations

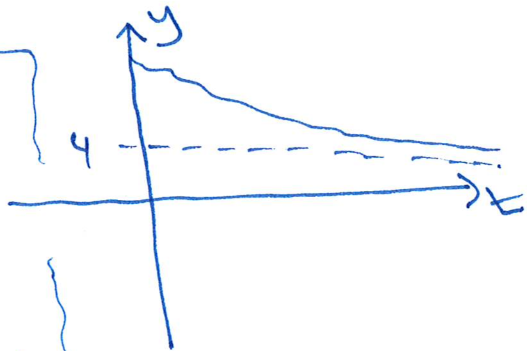
Ex: $y' = 0.05(4-y)$ — unknown function $y(t)$

Newton's law of cooling:

$y' = k(T-y)$ $y(t)$ temp. (°C) after t mins

$y > T: \; y' < 0$ T temp (°C) in the surroundings

k: constant



Defn: A differential equation is an equation that involves the derivative (and possibly higher derivatives) of the unknown function.

Ex:

$y' = 2t-2$ $ty' = y-1$

$y' = 0.05(4-y)$

$y' = y+t$

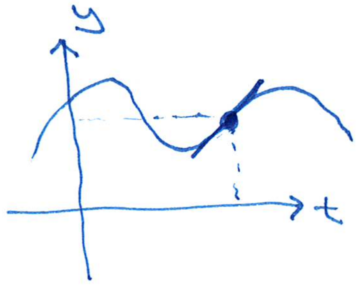
first order differential equations: involves y' but not higher derivatives

$y'' - 5y' + y = t$

$y'' = y \cdot y'$

second order differential equations

First order differential equations: $y' = F(t,y)$

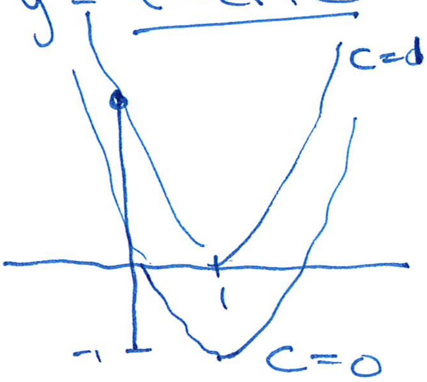


Ex: Simple integration

$y' = 2t-2 \quad | \int \ldots dt$

$y = \int 2t-2 \, dt = t^2 - 2t + C$

$y = \underline{t^2-2t+C}$ general solution



$\underbrace{y' = 2t-2}_{\text{diff. eqn.}}, \; \underbrace{y(0)=1}_{\text{initial condition}}$:

$y' = 2t-2 \Rightarrow \ldots \Rightarrow y = \underline{t^2-2t+C}$ general sol'n.

$\underline{y(0)=1}$: $1 = 0^2 - 2\cdot 0 + C \quad C=1$

$\Rightarrow \underline{y = t^2-2t+1}$

particular sol'n.

General fact:
Any first order differential equation has a general solution that depends on one undetermined coeff.

(2) Separable differential equations

Defn A first order differential eqn. is called separable if it can be written

$$\boxed{y' = f(t)\cdot g(y)}$$

Ex: $y' = ty$ separable $f(t) = t \quad g(y) = y$

$y' = t + y$ not separable

$ty' = 2y - 4 \Rightarrow y' = \frac{2y-4}{t} = \frac{2(y-2)}{t} = \underbrace{\frac{2}{t}}_{f(t)} \cdot \underbrace{(y-2)}_{g(y)}$

Separable

Method (solving separable diff. eqn.)

$y' = f(t)\cdot g(y) \quad | : g(y)$

$\frac{1}{g(y)} y' = f(t) \quad | \int \ldots dt$

$\int \frac{1}{g(y)} y' \, dt = \int f(t)\, dt$

$\int \frac{1}{g(y)} \, dy = \int f(t)\, dt$

"Substitution"

$$\boxed{\begin{gathered} y = y(t) \\ y'\,dt = dy \end{gathered}}$$

$y' = \frac{2}{t}(y-2)$

$\frac{1}{y-2} y' = \frac{2}{t}$

$\int \frac{1}{y-2} dy = \int \frac{2}{t} dt$

$\ln|y-2| + C_1 = 2\ln|t| + C_2$

$\ln|y-2| = 2\ln|t| + (C_2 - C_1)$

$\ln|y-2| = 2\ln|t| + C$

general solution in implicit form

To find an explicit form:

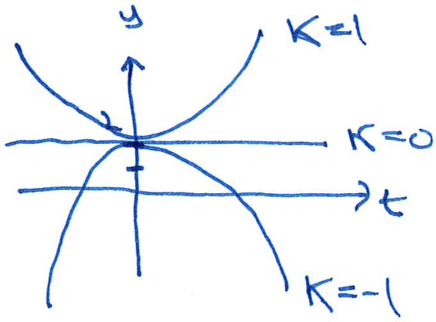
$$\ln|y-2| = 2\ln|t| + C \quad | e^{\cdot}$$

$$e^{\ln|y-2|} = e^{2\ln|t|+C}$$

$$|y-2| = e^{\ln|t|^2}\cdot e^C = |t|^2\cdot e^C = t^2\cdot e^C$$

$$y-2 = \pm t^2 e^C = (\pm e^C)\cdot t^2 = Kt^2$$

$$y = \underline{\underline{2+Kt^2}}$$ general solution in explicit form



Ex: $y' = ky$

$$\frac{1}{y}y' = k$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln|y| = kt + C \quad | e^{\cdot}$$

$$|y| = e^{kt+C}$$

$$y = \pm e^{kt}\cdot e^C = Ke^{kt}$$

$$y = \underline{\underline{Ke^{kt}}} = \underline{\underline{y_0\cdot e^{kt}}}$$

$$y(0) = K\cdot e^{k\cdot 0} = K$$

Ex: $y' = 0.05(4-y)$

$$\frac{1}{4-y}y' = 0.05$$

$$\int \frac{1}{4-y}\,dy = \int 0.05\,dt$$

$$-\ln|4-y| = 0.05t + C$$

$$\ln|4-y| = -0.05t - C$$

$$|4-y| = e^{-0.05t-C}$$

$$4-y = \pm e^{-0.05t}\cdot e^{-C} = Ke^{-0.05t}$$

$$4 = y + Ke^{-0.05t}$$

$$\underline{y = 4 - Ke^{-0.05t}}$$ general solution

$\underline{y(0) = 16.5}$: $16.5 = 4 - K\cdot e^{-0.05\cdot 0}$

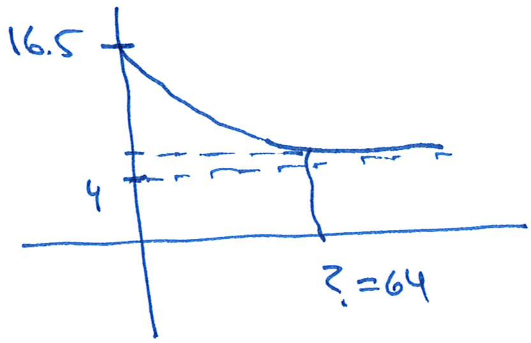
$$16.5 = 4 - K$$

$$K = 4 - 16.5 = -12.5$$

$$y(t) = 4 + 12.5e^{-0.05t}$$

Particular solution:

$$y = 4 + 12.5\,e^{-0.05t}$$



$y(t) = 4.5$:

$$4.5 = 4 + 12.5\,e^{-0.05t}$$
$$0.5 = 12.5\,e^{-0.05t} \quad | :12.5$$
$$e^{-0.05t} = \frac{0.5}{12.5} = \frac{1}{25} = 0.04$$
$$\ln(e^{-0.05t}) = \ln(0.04)$$
$$-0.05t = \ln(0.04)$$
$$t = \frac{\ln(0.04)}{-0.05} \simeq 64 \text{ min}$$

③ Linear first order differential equations

Defn: A first order diff. eqn. is linear if it can be written

$$\boxed{y' + a(t)\,y = b(t)} \iff y' = \underbrace{b(t) - a(t)\cdot y}_{\text{linear in } y}$$

Ex: $y' = y + t$ linear, not separable

$y' - y = t$ $\quad a(t) = -1 \quad b(t) = t$

Method (solving linear diff. eqn's)

$$y' + a(t)\,y = b(t) \quad | \cdot u(t)$$
$$u(t)\,y' + a(t)\,u(t)\,y = b(t)\cdot u(t)$$
$$u(t)\cdot y' + u'(t)\cdot y = b(t)\cdot u(t)$$
$$(u(t)\cdot y)' = b(t)\cdot u(t) \quad | \textstyle\int - dt$$
$$u(t)\cdot y = \int b(t)\,u(t)\,dt \implies \boxed{y = \frac{1}{u(t)}\int b(t)\,u(t)\,dt}$$

Integrating factor: $u(t)$

Want: $u' = a(t)\cdot u$

Need: $\boxed{u(t) = e^{\int a(t)\,dt}}$

Check: $u' = e^{\int a(t)\,dt}\cdot a(t) = u\cdot a(t)$

Summary:

To solve $y' + a(t)y = b(t)$ $\longrightarrow$ $\begin{cases} \text{Integrating factor: } u(t) = e^{\int a(t)\,dt} \\ \text{General solution: } y = \frac{1}{u(t)} \int u(t) b(t)\, dt \end{cases}$

Ex: $y' = y + t \Rightarrow y' - y = t \quad | \cdot e^{-t}$

$a(t) = -1 \quad b(t)$

Integrating factor:

$\int a(t)\,dt = \int -1\,dt = -t + C$

$u = e^{-t+C} = \underline{e^{-t}}$

$(e^{-t} \cdot y)' = te^{-t}$

$e^{-t}y = \int te^{-t}\,dt$

$u = -e^{-t} \quad v = t$

$u' = e^{-t} \quad v' = 1$

$= -e^{-t} \cdot t - \int e^{-t} \cdot 1\,dt$

$= -te^{-t} - (-e^{-t}) + C$

$e^{-t} \cdot y = -te^{-t} + e^{-t} + C \quad | \cdot e^{t}$

Int. by parts:

$\int u'v\,dt = uv - \int uv'\,dt$

$y = \underline{\underline{-t + 1 + Ce^{t}}}$

Ex: $ty' = 2y - 4 \quad | : t \qquad y' = \frac{2y-4}{t} = \frac{2}{t} \cdot (y-2)$ sep. ✓

$\boxed{y' - \frac{2}{t}y = -\frac{4}{t}} \qquad y' = \frac{2}{t} \cdot y - \frac{4}{t}$ ln ✓

linear eq.

$a(t) = -2/t \quad b(t) = -4/t$

Int. factor: $\int a(t)\,dt = \int -2/t\,dt = -2\ln|t| + C$

$u = e^{-2\ln|t| + C} = e^{-2\ln|t|} = e^{\ln|t|^{-2}} = |t|^{-2} = \frac{1}{|t|^2} = 1/t^2$

$$y' - \frac{2}{t} y = -\frac{4}{t} \quad \Big| \cdot \frac{1}{t^2}$$

$$\left(\frac{1}{t^2} \cdot y\right)' = -\frac{4}{t^3}$$

$$\frac{1}{t^2} y = \int -\frac{4}{t^3}\, dt = \int -4t^{-3}\, dt = -4\, \frac{t^{-2}}{-2} + C$$

$$1/t^2 \cdot y = 2 \cdot 1/t^2 + C \quad \Big| \cdot t^2$$

$$y = \underline{\underline{2 + Ct^2}}$$

④ Exact differential equations

Defn: A first order diff. eqn. is called exact if it can be written

$$\boxed{p(t,y) + q(t,y) \cdot y' = 0}$$

such that $\boxed{p(t,y) = h'_t \text{ and } q(t,y) = h'_y}$ for some function $h(t,y)$.

exactness cond.

Method: (exact diff. eqn.)

General solution: $h(t,y) = C$

Ex: $t^2 + y^2 y' = 0 \Rightarrow y^2 y' = -t^2 \Rightarrow y' = -\frac{t^2}{y^2} = -t^2 \cdot \frac{1}{y^2}$

$p = t^2$

$q = y^2$

$$\boxed{\begin{aligned} h'_t &= t^2 \\ h'_y &= y^2 \end{aligned}}$$

① $h = \int t^2\, dt = \underline{\frac{1}{3}t^3 + C(y)}$

② $\left(\frac{1}{3}t^3 + C(y)\right)'_y = y^2 \quad 0 + C'(y) = y^2$

Conclusion:

$h(t,y) = \frac{1}{3}t^3 + \frac{1}{3}y^3 + C$

is a solution to (1) and (2).

$\Rightarrow$ the diff. eqn. is exact

$C'(y) = y^2$

$C(y) = \int y^2 dy = \frac{1}{3}y^3 + C$

Solution: $h(t,y) = C$

$\frac{1}{3}t^3 + \frac{1}{3}y^3 + C_1 = C_2$

$\frac{1}{3}t^3 + \frac{1}{3}y^3 = C$

$\frac{1}{3}y^3 = C - \frac{1}{3}t^3 \quad | \cdot 3$

$y^3 = 3C - t^3$

$y = \sqrt[3]{3C - t^3}$