

Plan

- 1 Equilibrium states and stability
- 2 Linear second order differential equations

Integration: App. B
 i) Substitution or
 ii) int. by parts Lecture 4
 iii) partial fraction (FOREKØRS)

Review: First order differential equations

(1) Separable: $y' = f(t) \cdot g(y)$ $\frac{1}{g(y)} y' = f(t)$

$$\int \frac{1}{g(y)} dy = \int f(t) dt$$

(2) Linear: $y' + a(t)y = b(t)$ $u = e^{\int a(t) dt}$
 int. factor
 $y' = b(t) - a(t)y$

$$(y \cdot u)' = b(t) \cdot u(t)$$

$$y \cdot u = \int b(t) u(t) dt$$

$$y = \frac{1}{u(t)} \int b(t) u(t) dt$$

(3) Exact: $p(t,y) + q(t,y) y' = 0$
 with $p = h'_t$, $q = h'_y$

$$h(t,y) = C$$

Ex.: $y' = \frac{3t^2 - y^2}{2 + y}$ not separable
 not linear

$$2 + y \cdot y' = 3t^2 - y^2$$

$$\underbrace{y^2 - 3t^2}_p + \underbrace{2 + y \cdot y'}_q = 0$$

(1) $h'_t = y^2 - 3t^2$

$$h = y^2 t - t^3 + Q(y)$$

(2) $h'_y = 2 + y$

$$h'_y = 2yt - 0 + Q'(y) = 2 + y$$

Concl: Exact $h = y^2 t - t^3 = C$

$$Q'(y) = 0$$

$$Q(y) = 0$$

Why? $h = h(t,y)$
 $y = y(t)$
 Total derivative: $\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt}$
 $= h'_t + h'_y \cdot y'$
 $\frac{dh}{dt} = 0 \Rightarrow h = C$

$$\frac{y^2 t}{t} = \frac{t^3 + C}{t}$$

$$y^2 = \frac{t^3 + C}{t}$$

$$y = \pm \sqrt{\frac{t^3 + C}{t}}$$

① Equilibrium states and stability

Defn: A first order differential eqn. is called autonomous if it can be written $y' = F(y)$.

Ex: $y' - 4y = 2$ autonomous $y' - 4ty = 4t$ not
 $y' = 5(1 - y/10)$ — " — auton.

Defn: An equilibrium state of an auton. diff.-eq is a value $y = y_e$ such that $F(y_e) = 0$.

Ex: $y' - 4y = 2$

Eq. states: $4y + 2 = 0$

$y' = 4y + 2$

"constant solution"

$y = -2/4 = -1/2$

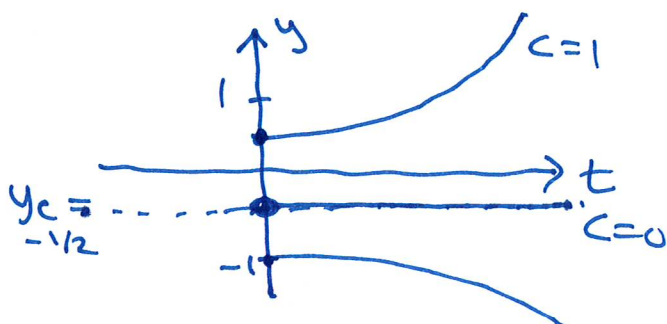
$y_e = -1/2$

Solve: $y' - 4y = 2 \quad | \cdot e^{-4t}$

$u = e^{-4t}$ $(ye^{-4t})' = 2e^{-4t}$

$ye^{-4t} = \int 2e^{-4t} dt = 2 \cdot \frac{1}{-4} e^{-4t} + C = -\frac{1}{2} e^{-4t} + C$

$y = e^{4t} \left(-\frac{1}{2} e^{-4t} + C \right) = -\frac{1}{2} + C e^{4t}$



$C=0: y(0) = -\frac{1}{2}$

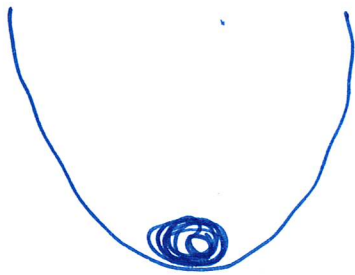
$C=1: y(0) = -\frac{1}{2} + 1 = \frac{1}{2}$

$y = -\frac{1}{2} + e^{4t}$

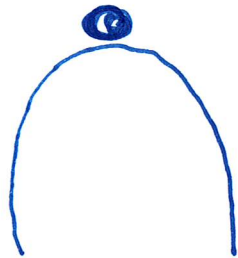
Defn: $y = y_e$ is a stable eq. state if y_0 close to y_e implies that $y(t) \rightarrow y_e$.

$y = y_e$ is a unstable eq. state if y_0 close to y_e implies that $y(t)$ moves away from y_e

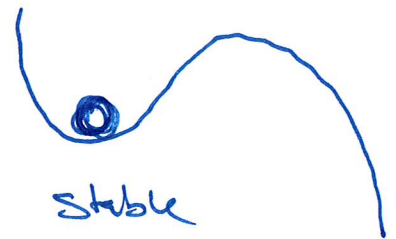
$y = y_e$ is globally asymptotically stable if $y(t) \rightarrow y_e$ no matter what y_0 you choose



Stable
glob. asympt. stable



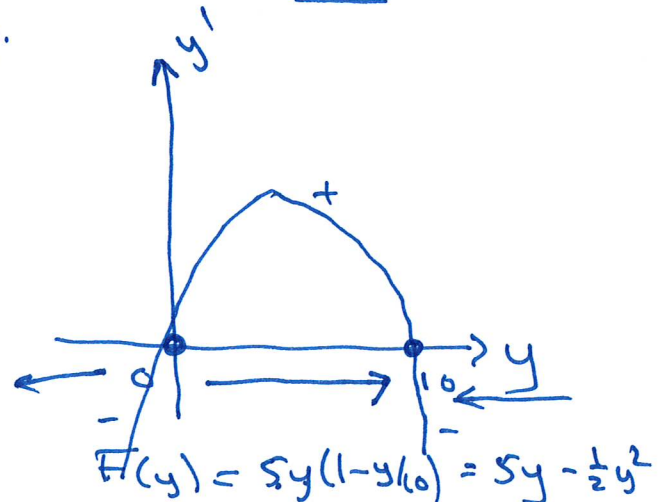
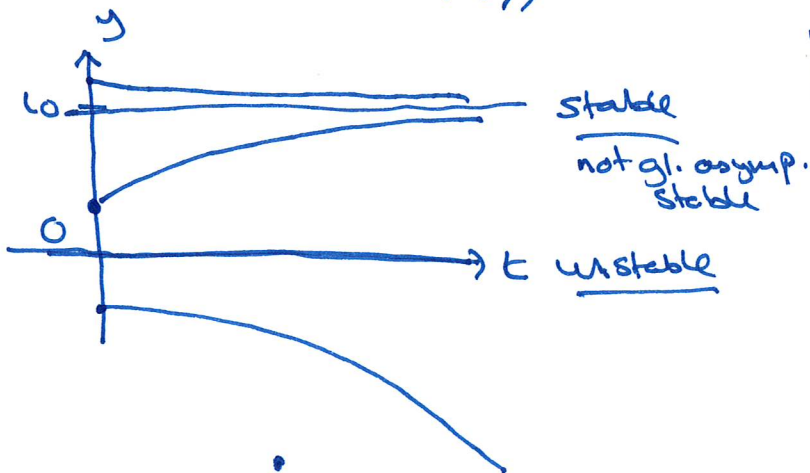
unstable



Stable
not globally asympt. stable

Ex: $y' = \underbrace{5y(1-y/10)}_{F(y)}$

Eq. states: $y_e = 0, y_e = 10$
 $5y(1-y/10) = 0$
 $y = 0$ or $1 - y/10 = 0$
 $y = 10$



Concl: $y_e = 0$ unstable
 $y_e = 10$ stable

Theorem (Stability)

If y_e is a eq. state of $y' = F(y)$ then we have:

$$1) F'(y_e) < 0 \Rightarrow y_e \text{ stable}$$

$$2) F'(y_e) > 0 \Rightarrow y_e \text{ unstable}$$

Ex: $y' = 5y(1 - y/10)$

$$F(y) = 5y(1 - y/10)$$

$$y_e = 0, y_e = 10$$

$$= 5y - \frac{1}{2}y^2$$

$$F'(0) = 5$$

$$F'(10) = -5$$

$$F'(y) = 5 - y$$

unstable

stable

② Second order linear differential equations

Ex: $y'' = 12t$ $y' = \int 12t dt = 6t^2 + C$

$$y = \int 6t^2 + C dt = \underline{\underline{2t^3 + Ct + D}}$$

A second order diff. eqn contains y'' but not higher derivatives. Its general solution depends on two undetermined coefficients.

Defn. A second order diff. eqn. is linear if it can be written

$$\boxed{y'' + a(t)y' + b(t)y = c(t)}$$

It is called homogeneous if $c(t) = 0$, and inhomogeneous otherwise.

It has constant coefficients if $a(t) = a$ and $b(t) = b$ are constants.

Ex: $y'' - 5y' + 6y = te^t$ const. coefft, inhom.
 $y'' - 3y' - 2y = 0$ ——— homog.

a) The homogeneous case: $y'' + ay' + by = 0$

Ex: $y'' - 3y' + 2y = 0$

$$\begin{cases} y = e^{rt} \cdot 1 \text{ (r parameter)} \\ y' = e^{rt} \cdot r = re^{rt} \\ y'' = (e^{rt} \cdot r) \cdot r = r^2 e^{rt} \end{cases}$$

$$r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0 \quad \leftarrow$$

$$e^{rt} \cdot (r^2 - 3r + 2) = 0$$

Char eqn: $r^2 - 3r + 2 = 0$

$$r = \frac{3 \pm \sqrt{9 - 8}}{2} = \frac{3 \pm 1}{2}$$

$$r_1 = \underline{2}, \quad r_2 = \underline{1}$$

General solution: $y = \underline{C_1 \cdot e^{2t} + C_2 e^t}$

In general:

$$e^{rt} (r^2 + ar + b) = 0$$

Char. eqn: $\boxed{r^2 + ar + b = 0}$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

i) $a^2 - 4b > 0: r_1 \neq r_2$
 $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Ex: $y'' - 4y' + 4y = 0$

Char. eqn: $r^2 - 4r + 4 = 0$
 $(r-2)^2 = 0$

$r_1 = r_2 = 2$

$\Rightarrow y = \underline{\underline{C_1 e^{2t} + C_2 t e^{2t}}}$

ii) $a^2 - 4b = 0$: $r_1 = r_2 = -a/2$
 $y = C_1 e^{-a/2 \cdot t} + C_2 t e^{-a/2 \cdot t}$

iii) $a^2 - 4b < 0$: need to compute with complex numbers

$$\begin{aligned} r &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} \\ &= -\frac{a}{2} \pm \frac{1}{2} \sqrt{4b - a^2} \cdot \sqrt{-1} \\ &= \alpha \pm \beta \cdot \sqrt{-1} \\ y &= e^{\alpha t} \cdot (C_1 \cos(\beta t) + C_2 \sin(\beta t)) \end{aligned}$$

b) The inhomogeneous case:

$y'' + ay' + by = c(t)$

Ex: $y'' - 5y' + 6y = 4e^{-t}$

$y = y_h + y_p = \underline{\underline{C_1 e^{2t} + C_2 e^{3t} + \frac{1}{3} e^{-t}}}$

y_h : $y'' - 5y' + 6y = 0$

Char. eqn: $r^2 - 5r + 6 = 0$

$\underline{r=2}, \underline{r=3}$

$y_h = C_1 e^{2t} + C_2 e^{3t}$

y_p : $y'' - 5y' + 6y = 4e^{-t}$

Superposition principle

General solution of a linear diff. eqn. is

$y = y_h + y_p$

where

y_h is the general soln to the homogeneous eqn.

y_p is a particular solution to the nonhom. eqn.

$$\underline{c(t) = 4e^{-t}}; \quad y = A \cdot e^{-t}$$

$$c'(t) = 4e^{-t} \cdot (-1) \quad y' = A \cdot e^{-t} \cdot (-1) = -Ae^{-t}$$

$$c''(t) = 4e^{-t} \cdot (-1)^2 \quad y'' = -Ae^{-t} \cdot (-1) = Ae^{-t}$$

$$\underline{y'' - 5y' + 6y = 4e^{-t}}; \quad (Ae^{-t}) - 5(-Ae^{-t}) + 6(Ae^{-t}) = 4e^{-t}$$

$$Ae^{-t} + 5Ae^{-t} + 6Ae^{-t} = 4e^{-t}$$

$$(12A)e^{-t} = 4e^{-t}$$

Compare coeff: $12A = 4$

$$A = 4/12 = \underline{\underline{1/3}}$$

$$\underline{y_p = \frac{1}{3}e^{-t}}$$

Notice:

$$\boxed{y'' - 5y' + 6y = 4e^{-t}}$$

$$D(y) = 4e^{-t}$$

where

$$D = \frac{d^2}{dt^2} + 5\frac{d}{dt} + 6$$

differential
operator

$$y_h \rightarrow D(c_1e^{2t} + c_2e^{3t}) = 0$$

$$y_p \rightarrow D\left(\frac{1}{3}e^{-t}\right) = 4e^{-t}$$

"

$$D(y) = y'' - 5y' + 6y$$

$$D\left(\underbrace{c_1e^{2t} + c_2e^{3t}}_{y_h} + \underbrace{\frac{1}{3}e^{-t}}_{y_p}\right) = 0 + 4e^{-t} = 4e^{-t}$$

Ex: $y'' - 3y' + 2y = t + 4$

$$y = y_h + y_p = \underline{C_1 e^t + C_2 e^{2t} + \frac{1}{2}t + \frac{11}{4}}$$

y_h : $y'' - 3y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$

$$\underline{r=1}, \underline{r=2}$$

$$\Rightarrow y_h = C_1 e^t + C_2 e^{2t}$$

y_p : $y'' - 3y' + 2y = t + 4$

$$c(t) = t + 4$$

$$c'(t) = 1$$

$$c''(t) = 0$$

$$y = \underline{At + B}$$

$$y' = A$$

$$y'' = 0$$

$$0 - 3A + 2(A + B) = t + 4$$

$$\underbrace{(2A)t}_{=1} + \underbrace{(-3A + 2B)}_{=4} = t + 4$$

$$2A = 1$$

$$-3A + 2B = 4$$

$$A = \underline{1/2}$$

$$-3(1/2) + 2B = 4$$

$$2B = 4 + 3/2 = 11/2$$

$$B = \underline{11/4}$$

$$y_p = \underline{\frac{1}{2}t + \frac{11}{4}}$$

Ex: $y'' - 3y' + 2y = t^2$

y_p : $C(t) = t^2$
 $C'(t) = 2t$
 $C''(t) = 2$

y_h : $r^2 - 3r + 2 = 0$
 $r=1, r=2$

$y_h = C_1 e^t + C_2 e^{2t}$

$y = \underline{At^2 + Bt + C}$
 will work

y_p : $2A - 3(2At + B) + 2(A t^2 + Bt + C) = t^2$ ←
 $(2A)t^2 + (-6A + 2B)t + (2A - 3B + 2C) = t^2$
 $\begin{matrix} \text{"} & & \text{"} & & \text{"} \\ 1 & & 0 & & 0 \end{matrix}$

$\left\{ \begin{array}{l} y = At^2 + Bt + C \\ y' = 2At + B \\ y'' = 2A \end{array} \right.$

$2A = 1 \quad A = \underline{1/2}$

$-6A + 2B = 0 \quad -6(1/2) + 2B = 0 \quad 2B = 3 \quad B = \underline{3/2}$

$2A - 3B + 2C = 0 \quad 2(1/2) - 3(3/2) + 2C = 0 \quad 2C = 9/4 - 1 = 5/4 \quad C = \underline{5/8}$

$y_p = \underline{\underline{\frac{1}{2}t^2 + \frac{3}{2}t + \frac{5}{8}}}$

$y = y_h + y_p = \underline{\underline{C_1 e^t + C_2 e^{2t} + \frac{1}{2}t^2 + \frac{3}{2}t + \frac{5}{8}}}$

(Since there are no soln with $B=C=0$,
 we see that $y = At^2$ would not have worked)

Ex: $y'' - 3y' + 2y = 4e^{2t}$

$$y = y_h + y_p = c_1 e^t + c_2 e^{2t} + \dots$$

y_h : $r^2 - 3r + 2 = 0$ $y_h = c_1 e^t + c_2 e^{2t}$
 $r = 1, r = 2$

y_p : $y'' - 3y' + 2y = 4e^{2t}$

$$c(t) = 4e^{2t}$$

$$c' = 8e^{2t}$$

$$c'' = 16e^{2t}$$

① $y = Ae^{2t}$ $\left\{ \begin{array}{l} y' = 2Ae^{2t} \\ y'' = 4Ae^{2t} \end{array} \right.$

$$4Ae^{2t} - 3(2Ae^{2t}) + 2(Ae^{2t}) = 4e^{2t}$$

$$(4A - 6A + 2A)e^{2t} = 4e^{2t}$$

$0 \cdot A = 4$
no soln

$y = Ae^{2t}$

will not work

Try: $y = Ate^{2t}$

② $y = Ate^{2t}$
 $y' = Ae^{2t} + At \cdot e^{2t} \cdot 2 = (A + 2At)e^{2t}$

$$y'' = 2Ae^{2t} + (A + 2At)e^{2t} \cdot 2 = (4A + 4At)e^{2t}$$

$$(4A + 4At)e^{2t} - 3(A + 2At)e^{2t} + 2(At)e^{2t} = 4e^{2t}$$

$$(4A + 4At - 3A - 6At + 2At)e^{2t} + (4A - 3A)e^{2t} = 4e^{2t}$$

$0 \cdot At$

$$Ae^{2t} = 4e^{2t}$$

$A = 4$

$$y_p = 4te^{2t}$$

$$y = y_h + y_p = \underline{\underline{c_1 e^t + c_2 e^{2t} + 4te^{2t}}}$$