Plan

1. Systems of linear first order differential equations
2. Equilibrium states and stability

Plenary Session 4

Lecture 10-12

Monday 13/11

① Systems of differential equations

Ex: $y_1' = y_1 - 3y_2$

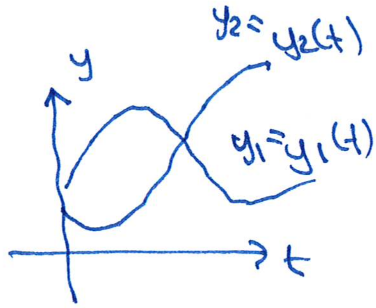
$y_2' = -3y_1 + y_2$

"coupled"

unknown functions

$y_1 = y_1(t)$

$y_2 = y_2(t)$



Defn: A system of linear first order autonomous differential equations is one that can be written

$$\boxed{\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n + b_1 \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n + b_2 \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n + b_n \end{aligned}}$$

where a_{ij}, b_i are constants.

$\underline{y}' = A \cdot \underline{y} + \underline{b}$

matrix form

$$\underline{y}' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

square $(n \times n)$ matrix $\longrightarrow$ $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Ex: $y_1' = 4y_1$, $y_2' = -2y_2$ | $y_1' = y_1 - 3y_2$, $y_2' = -3y_1 + y_2$ | $y_1' = y_2$, $y_2' = -y_1$

$\underline{y}' = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \underline{y}$ | $\underline{y}' = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \underline{y}$ | $\underline{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{y}$

"the good" | "the bad" | "the ugly"

$\rightarrow$ decoupled = diagonal matrix | not diagonalizable $\lambda^2+1=0$

$\underline{y_1' = 4y_1}$:

$y_1' - 4y_1 = 0$ lin. homog.

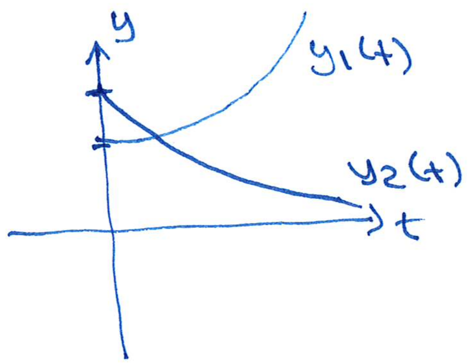
$r - 4 = 0$ $r = 4$ $\underline{y_1 = C_1 e^{4t}}$

$\underline{y_2' = -2y_2}$:

$y_2' + 2y_2 = 0$ lin. homog.

$r + 2 = 0$ $r = -2$ $\underline{y_2 = C_2 e^{-2t}}$

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{4t} \\ C_2 e^{-2t} \end{pmatrix}$$



i) The homogeneous case: $\underline{y}' = A\underline{y}$

We assume that A is diagonalizable, this means:

$P^{-1}AP = D$ where $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$ and $P = \begin{pmatrix} \underline{v_1} \,|\, \underline{v_2} \,|\, \cdots \,|\, \underline{v_n} \end{pmatrix}$

and $A \cdot \underline{v_i} = \lambda_i \underline{v_i}$ and $\underline{v_1}, \ldots, \underline{v_n}$ lin. independent.

Change of variables: $\underline{y} = P\underline{z} \Leftrightarrow \underline{z} = P^{-1}\underline{y}$

$\underline{y}' = A\underline{y}$: $\underline{y}' = P\underline{z}'$

$P\underline{z}' = A \cdot P\underline{z}$ $\mid P^{-1}\cdot$

$\underline{z}' = (P^{-1}AP)\underline{z} = D\underline{z}$

new system

$\underline{z}' = D \cdot \underline{z}$

$\underline{y} = P\underline{z}$ means

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$y_1 = p_{11} z_1 + \ldots + p_{1n} z_n$

Ex: $\underline{y}' = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \underline{y}$

Diagonalization:

Eigenvalues:

$$\lambda^2 - 2\lambda - 8 = 0$$
$$(\lambda - 4)(\lambda + 2) = 0$$
$$\underline{\lambda_1 = 4}, \quad \underline{\lambda_2 = -2}$$

Eigenvectors:

E_4: $\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}$

$-3x - 3y = 0$, y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

E_{-2}: $\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$ $\quad 3x - 3y = 0$, y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$

$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

Change of variables:

$$\underline{y} = P \cdot \underline{z} \iff \underline{z} = P^{-1} \underline{y}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} y_1 &= -z_1 + z_2 \\ y_2 &= z_1 + z_2 \end{aligned}}$$

$$\boxed{\begin{aligned} z_1 &= \tfrac{1}{2}(-y_1 + y_2) \\ z_2 &= \tfrac{1}{2}(y_1 + y_2) \end{aligned}}$$

$\Updownarrow$

$\underline{z}' = D \underline{z}$ $\qquad z_1' = 4z_1$, $\quad z_2' = -2z_2$

$\underline{z} = \begin{pmatrix} c_1 e^{4t} \\ c_2 e^{-2t} \end{pmatrix}$ $\qquad z_1 = c_1 e^{4t}$, $\quad z_2 = c_2 e^{-2t}$

$\Updownarrow$

$y_1(t)$

$$\underline{y} = P \cdot \underline{z} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 e^{4t} \\ c_2 e^{-2t} \end{pmatrix} = \begin{pmatrix} -c_1 e^{4t} + c_2 e^{-2t} \\ c_1 e^{4t} + c_2 e^{-2t} \end{pmatrix}$$

$y_2(t)$

$$\| $$

$$(\underline{v}_1 | \underline{v}_2) \cdot \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} = \underline{v}_1 \cdot c_1 e^{\lambda_1 t} + \underline{v}_2 \cdot c_2 e^{\lambda_2 t}$$
$$= \underline{c_1 \cdot \underline{v}_1 e^{\lambda_1 t} + c_2 \cdot \underline{v}_2 e^{\lambda_2 t}} = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{-2t}$$

Result: When A is a diagonalizable $n \times n$-matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with mult.) and lin. independent eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$ such that $A\underline{v}_i = \lambda_i \underline{v}_i$, then $\underline{y}' = A\underline{y}$ has general solution:

$$\boxed{\underline{y} = C_1 \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} + \ldots + C_n \underline{v}_n e^{\lambda_n t}}$$

ii) The non-homogeneous case: $\underline{y}' = A\underline{y} + \underline{b}$

Assume A is diagonalizable

Ex:
$$y_1' = y_1 - 3y_2 + 4$$
$$y_2' = -3y_1 + y_2 + 2$$

$$\underline{y}' = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \underline{y} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

We can use the superposition principle:

General solution: $\underline{y} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 5/4 \\ 7/4 \end{pmatrix}$

$$\underline{y} = \underline{y}_h + \underline{y}_p$$

$$\underline{y}_h = C_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + C_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$$

$\underline{y}_h$: general solution of the homogeneous system of diff. eqn's. $\underline{y}' = A\underline{y}$

$\underline{y}_p$:
$$\underline{y}' = A\underline{y} + \underline{b}$$
$$\underline{0} = A\underline{y} + \underline{b}$$
$$A\underline{y} = -\underline{b}$$

$$\underline{y}_p = \begin{pmatrix} 5/4 \\ 7/4 \end{pmatrix}$$

$\underline{y}_p$: particular solution of $\underline{y}' = A\underline{y} + \underline{b}$

→ look for constant solutions
$$0 = A \cdot \underline{y} + \underline{b}$$
$$A\underline{y} = -\underline{b}$$

$$\begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \underline{y} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} : \left(\begin{array}{cc|c} 1 & -3 & -4 \\ -3 & 1 & -2 \end{array}\right) \rfloor 3$$

$$\to \left(\begin{array}{cc|c} 1 & -3 & -4 \\ 0 & -8 & -14 \end{array}\right) \quad \begin{aligned} y_1 - 3(7/4) &= -4 \Rightarrow y_1 = \tfrac{21}{4} - \tfrac{16}{4} = 5/4 \\ -8y_2 &= -14 \Rightarrow y_2 = \tfrac{14}{8} = 7/4 \end{aligned}$$

Comment: Second order linear differential equations that are autonomous.

$$y'' + ay' + by = c$$

Ex: $y'' - 3y' + 2y = 12 \to y'' = 3y' - 2y + 12$

$y = y_h + y_p$

y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 2,\ r = 1$

$y_h = C_1 e^{2t} + C_2 e^{t}$

y_p: $y'' - 3y' + 2y = 12$

$y = A$: $0 - 3\cdot 0 + 2A = 12$

$y' = 0$ $\quad 2A = 12$

$y'' = 0$ $\quad A = 6$

$y_p = 6$

$y = C_1 e^{2t} + C_2 e^{t} + 6$

$y' = 2C_1 e^{2t} + C_2 e^{t}$

$y_1 = y$ $\quad y_1' = y' = y_2$

$y_2 = y'$ $\quad y_2' = y'' = -2y_1 + 3y_2 + 12$

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$:

Eigenvalues $\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 2,\ \lambda = 1$

Eigenvectors:

E_2: $\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \rightsquigarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

E_1: $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \rightsquigarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\mathbf{y}_h = C_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + C_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$

$\mathbf{y}_p$: $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y_2 = 0$ $\quad y_2 = 0$

$-2y_1 + 3y_2 = -12$ $\quad y_1 = \frac{-12}{-2} = 6$

$\mathbf{y}_p = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$\mathbf{y} = C_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + C_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

2 Equilibrium states and stability

(a) <u>System of diff. eqn:</u> $\underline{y}' = A\underline{y} + \underline{b}$

<u>Eq. states:</u> $\underline{y}' = \underline{0} \iff A\underline{y} + \underline{b} = \underline{0}$

$A\underline{y} = -\underline{b}$

if there is a particular solution $\underline{y}_p$ that is constant, then this is the <u>eq. state</u>.

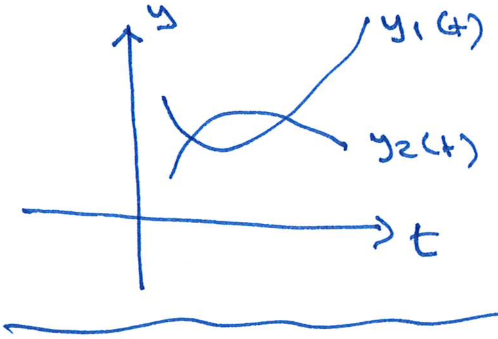
<u>Ex:</u> $\underline{y}' = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}\underline{y} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\qquad \underline{y}_p = \begin{pmatrix} 5/4 \\ 7/4 \end{pmatrix}$ const. sol.n

$\Rightarrow \underline{y}_e = \begin{pmatrix} 5/4 \\ 7/4 \end{pmatrix}$

<u>unstable</u>

<u>General solution:</u>

$$\underline{y} = \underbrace{c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}}_{\underline{y}_h} + \underbrace{\begin{pmatrix} 5/4 \\ 7/4 \end{pmatrix}}_{\underline{y}_p = \underline{y}_e}$$



<u>Result:</u>

$\lambda_1, \lambda_2, \ldots, \lambda_n < 0 \iff \underline{y}_e$ is stable (and gl. asymp. stable)

all other cases $\iff \underline{y}_e$ is unstable

(b) Second order diff. eqns: $y'' + ay' + by = c$

Eq. states: Constant solutions: $by = c$

$y' = 0 \quad y'' = 0$

$y_e = c/b \quad (b \neq 0)$

" y_p

$$y = y_h + y_p = \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}_{y_h} + \underbrace{c/b}_{y_p}$$

$r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$\uparrow$

r_1, r_2

Result:

y_e is stable $\iff$ $r_1, r_2 < 0$ (gl. asymp. stable)

y_e is unstable $\iff$ otherwise