Plan

1 Linear difference equations
2 Systems of linear first order difference equations
3 Equilibrium states and stability

← Notice for eq. states/stability → not in the curriculum this year.

1 Linear difference equations

Ex: $\underbrace{y_{t+1} = 1.04 y_t}_{\text{difference eqn.}}$, $\underbrace{y_0 = 100}_{\text{initial condition}}$

$y_1 = \underline{1.04 y_0} = 104$

$y_2 = 1.04 \cdot y_1 = 1.04(1.04 y_0) = \underline{1.04^2 y_0} = 108.16$

$y_3 = 1.04 y_2 = 1.04(1.04^2 y_0) = \underline{1.04^3 y_0}$

$\vdots$

$y_t = \underline{1.04^t \cdot y_0}$ $t = 0, 1, 2, \ldots$ general solution

$= \underline{100 \cdot 1.04^t}$ particular solution

Note: - the solutions of a difference equation are sequences $(y_0, y_1, y_2, \ldots, y_t, \ldots)$

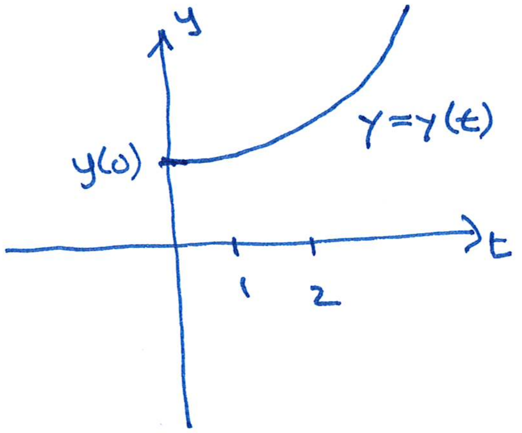
- we write y_t for sequences, as opposed to $y(t)$ for cont. functions - we only have values for $t = 0, 1, 2, \ldots$

- we can rewrite $y_{t+1} = 1.04 y_t$ as $\boxed{\underbrace{y_{t+1} - y_t}_{\text{difference in the seq.}} = 0.04 y_t}$

First order difference equation:
Eqn. with y_{t+1} and y_t

Compare:

$$y' = 0.04\,y$$
$$y' - 0.04y = 0$$
$$y = \underline{C \cdot e^{0.04t}} \quad e^{rt}$$
$$y(0) = C \cdot e^{0} = C$$
$$y = \underline{y(0)\, e^{0.04t}}$$

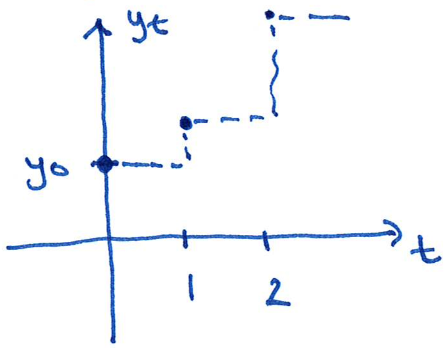


$$y_{t+1} - y_t = 0.04\,y_t$$
$$y_{t+1} = 1.04\,y_t$$
$$y_{t+1} - 1.04\,y_t = 0$$
$$y_t = \underline{C \cdot 1.04^t}$$
$$y_0 = C \cdot 1 = C$$
$$y_t = \underline{y_0 \cdot 1.04^t}$$

r = char. root

r^t or $(1+r)^t$

r = "interest"



Differential equations
= change in cont. time

Difference equations
= change in discrete time

Linear first order difference equations with constant coefficients.

$$\boxed{y_{t+1} + a y_t = b_t}$$ where a = constant, b_t = sequence

Ex: $y_{t+1} - 1.10 y_t = -300$ $\begin{cases} a = -1.10 \\ b = -300 \end{cases}$

Solution method: Superposition

$$y_t = y_t^h + y_t^p = \underline{C \cdot 1.10^t + 3000}$$

y_t^h: $y_{t+1} - 1.10 y_t = 0$

Char. eqn: $r - 1.10 = 0$

$\underline{r = 1.10}$ $\quad y_t^h = \underline{C \cdot 1.10^t}$

Char. eqn: $y_t = r^t$

$y_{t+1} = r^{t+1}$

$= r^t \cdot r = r \cdot r^t$

$r \cdot r^t - 1.10 \cdot r^t = 0$

$r^t \cdot (r - 1.10) = 0$

$\underline{r - 1.10 = 0}$

y_t^p: $y_{t+1} - 1.10 y_t = -300$

Guess: $y_t = A$ (constant soln) $\quad y_{t+1} = A$

$A - 1.10 A = -300$

$$\frac{-0.10 A}{-0.10} = \frac{-300}{-0.10}$$

$A = \underline{3000}$ $\quad y_t^p = \underline{3000}$

Interpretation:

$y_{t+1} - 1.10 y_t = -300$

$y_{t+1} - y_t = 0.10 y_t - 300$

bank acc. with interest 10% and 300 withdrawal each term

$y_t = (y_0 - 3000) \cdot 1.10^t + 3000$

$= \underline{y_0 \cdot 1.10^t - 3000(1 - 1.10^t)}$

$y_t = C \cdot 1.10^t + 3000$ $\quad t = 0$: $y_0 = C \cdot 1 + 3000$ $\quad C = y_0 - 3000$

Second order linear difference equations with const. coeff.

$$y_{t+2} + a y_{t+1} + b y_t = c_t \quad \text{where} \begin{cases} a, b & \text{constants} \\ c_t & \text{seq.} \end{cases}$$

Ex: $y_{t+2} - 3y_{t+1} + 2y_t = 0$

Solution method: superposition

$$y_t = y_t^h + y_t^p = y_t^h = \underline{\underline{C_1 + C_2 \cdot 2^t}}$$

y_t^h: $y_{t+2} - 3y_{t+1} + 2y_t = 0$

$$r^2 - 3r + 2 = 0$$

$$\underline{r=1}, \underline{r=2} \longrightarrow y_t^h = C_1 \cdot 1^t + C_2 \cdot 2^t = \underline{C_1 + C_2 \cdot 2^t}$$

Ex: $y_{t+2} - 4y_{t+1} + 4y_t = 0$

$$y_t = \underline{\underline{(C_1 + C_2 t) \cdot 2^t}}$$

y_t^h: $r^2 - 4r + 4 = 0$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = \underline{2} \longrightarrow y_t^h = \underline{C_1 \cdot 2^t + C_2 t \cdot 2^t}$$

Ex: $y_{t+2} - 3y_{t+1} + 2y_t = 4t$ $\qquad c_t = 4t$

$$y_t = y_t^h + y_t^p = \underline{\underline{C_1 \cdot 1^t + C_2 \cdot 2^t + 2t^2 - 2t}} = \underline{\underline{C_1 + C_2 \cdot 2^t - 2t^2 - 2t}}$$

y_t^h: $r^2 - 3r + 2 = 0$

$r = 1, 2$ $\qquad y_t^h = \underline{C_1 \cdot 1^t + C_2 \cdot 2^t}$

$\underline{y_t^p}$: $y_{t+2} - 3y_{t+1} + 2y_t = 4t$

$$\cancel{y_t = At + B}$$
$$\cancel{y_{t+1} = A(t+1) + B = At + A + B}$$
$$\cancel{y_{t+2} = A(t+2) + B = At + 2A + B}$$

$$\cancel{(At + 2A + B) - 3(At + A + B) + 2(At + B) = 4t}$$
$$\cancel{At - 3At + 2At + (\ldots)}$$
$$\cancel{0 \cdot t + (\ldots)} = 4t$$

$$y_t = \underline{At^2 + Bt}$$
$$y_{t+1} = A(t+1)^2 + B(t+1) = \underline{At^2 + (2A+B)t + (A+B)}$$
$$y_{t+2} = A(t+2)^2 + B(t+2) = \underline{At^2 + (4A+B)t + (4A+2B)}$$

$$\cancel{At^2} + (4A+B)t + (4A+2B)$$
$$-3(\cancel{At^2} + (2A+B)t + (A+B))$$
$$+2(\cancel{At^2} + Bt) \qquad = 4t$$

$$(4A + B - 6A - 3B + 2B)t + (4A + 2B - 3A - 3B) = 4t + 0$$

$$\left.\begin{aligned} -2A &= 4 \qquad A = -2 \\ A - B &= 0 \qquad B = A = -2 \end{aligned}\right\} \quad y_t^p = \underline{-2t^2 - 2t}$$

Difference eqn. with y_{t+2}, y_{t+1}, y_t = second order difference eqn.

Why? $\Delta(y_t) = y_{t+1} - y_t$ ...

$$\Delta^2(y_t) = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t)$$

$$= y_{t+2} - 2y_{t+1} + y_t$$

② Systems of difference equations

Ex: $y_{t+1} = y_t - z_t$ or $\begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} y_t \\ z_t \end{pmatrix}$

$z_{t+1} = -y_t + 3z_t$

Solution methods: Homogeneous case

$\underline{y}_{t+1} = A\,\underline{y}_t$ has general solution $\underline{y}_t = C_1 \cdot \underline{v}_1 \cdot \lambda_1^t + \ldots + C_n \underline{v}_n \lambda_n^t$

when A is diagonalizable, with eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$ that are linearly independent.

Ex: $A = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$ Eigenvalues: $\lambda^2 - 4\lambda + 2 = 0$

$$\lambda = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \frac{\sqrt{8}}{2} = 2 \pm \sqrt{2}$$

$\underline{y}_{t+1} = A\underline{y}_t$: $\underline{y}_t = C_1 \cdot \underline{v}_1 \cdot (2+\sqrt{2})^t + C_2 \cdot \underline{v}_2 \cdot (2-\sqrt{2})^t$

Ex: $y_{t+1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} y_t$ $\qquad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, \lambda = 3$

$\lambda = 1$: $\left(\begin{array}{cc|} 1 & -1 \\ \cancel{-1} & \cancel{1} \end{array}\right)$ $\quad x - y = 0$

$x = y, \ y$ free

$y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 3$: $\left(\begin{array}{cc|} -1 & -1 \\ \cancel{-1} & \cancel{-1} \end{array}\right)$ $\quad -x - y = 0$

$x = -y, \ y$ free

$y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \underline{v_3} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

General solution: $y_t = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 1^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3^t$

$= c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3^t$

$$y_t = \begin{pmatrix} c_1 - c_2 \cdot 3^t \\ c_1 + c_2 \cdot 3^t \end{pmatrix}$$

Ex: $y_{t+1} = \begin{pmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{pmatrix} y_t$ $\qquad A = \begin{pmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{pmatrix}$

regular Markov chain matrix

Eigenvalues: $\lambda^2 - 1.7\lambda + 0.7 = 0$

$(\lambda - 1)(\lambda - 0.7) = 0$

$\lambda = 1, \ \lambda = 0.7$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 5x \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

E_1: $\begin{pmatrix} -0.25 & 0.05 \\ \cancel{0.25} & \cancel{-0.05} \end{pmatrix} \to \begin{pmatrix} -5 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad -5x + y = 0$ $\quad y = 5x$ $\quad \underline{v_1} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$\cancel{x}$ free

$E_{0.7}$: $\begin{pmatrix} 0.05 & 0.05 \\ \cancel{0.25} & \cancel{0.25} \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad x + y = 0$ $\quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

y free

<u>Solution:</u> $\underline{y}_t = c_1 \begin{pmatrix}1\\5\end{pmatrix} \cdot 1^t + c_2 \begin{pmatrix}-1\\1\end{pmatrix} 0.7^t$

$$= \underline{\underline{c_1 \cdot \begin{pmatrix}1\\5\end{pmatrix} + c_2 \cdot \begin{pmatrix}-1\\1\end{pmatrix} \cdot 0.7^t}}$$

<u>Ex:</u> $\underline{y}_{t+1} = A\underline{y}_t + \underline{b} = \begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix} \underline{y}_t + \begin{pmatrix}-1\\1\end{pmatrix}$

$\underline{y}_{t+1} - \begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix} \underline{y}_t = \begin{pmatrix}-1\\1\end{pmatrix}$ <u>inhomogeneous</u>

$$\underline{y}_t = \underline{y}_t^h + \underline{y}_t^p = \underline{\underline{c_1 \cdot \begin{pmatrix}1\\1\end{pmatrix} \cdot 1^t + c_2 \cdot \begin{pmatrix}-1\\1\end{pmatrix} \cdot 3^t + \begin{pmatrix}1\\0\end{pmatrix}}}$$

$\underline{y_t^h}$: $\underline{y}_t^h = c_1 \cdot \begin{pmatrix}1\\1\end{pmatrix} \cdot 1^t + c_2 \cdot \begin{pmatrix}-1\\1\end{pmatrix} \cdot 3^t$

since $\lambda_1 = 1, \lambda_2 = 3$ $\underline{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix}-1\\1\end{pmatrix}$ (see previous ex.)

$\underline{y_t^p}$: $\underline{y}_{t+1} - A\underline{y}_t = \begin{pmatrix}-1\\1\end{pmatrix}$ <u>Guess:</u> Const. solution $\underline{y}_t = \underline{w} = \begin{pmatrix}A\\B\end{pmatrix}$

$\underline{w} - A\underline{w} = \begin{pmatrix}-1\\1\end{pmatrix} \quad | \cdot (-1)$

$-\underline{w} + A\underline{w} = \begin{pmatrix}1\\-1\end{pmatrix}$

$A\underline{w} - \underline{w} = \begin{pmatrix}1\\-1\end{pmatrix}$

$(A - I)\underline{w} = \begin{pmatrix}1\\-1\end{pmatrix}$

$\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} \underline{w} = \begin{pmatrix}1\\-1\end{pmatrix}$

$\underline{y}_t^p = \begin{pmatrix}1\\0\end{pmatrix}$

$\left(\begin{array}{cc|c}1 & -1 & 1\\-1 & 1 & -1\end{array}\right) \rightarrow \left(\begin{array}{cc|c}1 & -1 & 1\\0 & 0 & 0\end{array}\right)$ $A - B = 1$, B free $\rightarrow$ <u>For example:</u> $B = 0 \Rightarrow A = 1$, $\underline{w} = \begin{pmatrix}1\\0\end{pmatrix}$

Summary: Linear difference equations

Cases:

(a) First order linear: $y_{t+1} + a y_t = b_t$

(b) Second order linear: $y_{t+2} + a y_{t+1} + b y_t = c_t$

Method: $y_t = y_t^h + y_t^p$

- where y_t^h is found using the characteristic eqn.

  (a) $r + a = 0 \quad r = -a \implies y_t^h = \underline{C \cdot (-a)^t}$

  (b) $r^2 + ar + b = 0$

  if $r_1 \neq r_2$: $\quad y_t^h = \underline{C_1 \cdot r_1^t + C_2 \cdot r_2^t}$

  if $r_1 = r_2$: $\quad y_t^h = \underline{C_1 \cdot r_1^t + C_2 \cdot t \cdot r_1^t}$

- and where y_t^p is one particular solution, that can be found by guessing a solution y_t with parameters and substitute into the eqn. to find the parameter values (remember to compute y_{t+1}, y_{t+2})

(c) Linear systems: $\underline{y}_{t+1} - A \underline{y}_t = \underline{b}$

Method: $\underline{y}_t = \underline{y}_t^h + \underline{y}_t^p$

- where $\underline{y}_t^h = C_1 \underline{v}_1 \lambda_1^t + \ldots + C_n \underline{v}_n \lambda_n^t$, with $\lambda_i, \underline{v}_i$ eigenvalues and eigenvectors of A

- and $\underline{y}_t^p$ is a constant solution $\underline{y}_t = \underline{w}$, that is

$$\underline{w} - A\underline{w} = \underline{b} \iff A\underline{w} + \underline{b} = \underline{w} \iff A\underline{w} - \underline{w} = -\underline{b}$$

$$(A - I)\underline{w} = -\underline{b}$$

Eivind Eriksen (eivind.eriksen@bi.no), Office B3y-085