Plan

1 About the exam and Review of the course
2 Final exam 11/2022 + 12/2021

TA sessions:
today 16-18 D1-065
Monday 14-16 A2 B1+4

Remember:
Course evaluation

1 About the exam

Topics: - linear algebra (matrices/vectors)
- max/min-problems
- differential/difference eqn's.

How to prepare: - previous exam problems (final and midterm)
- look at the lecture notes
- key problems

Fake exam	Final exam in GRA 6035 Mathematics
Date	November 28th 2023 at 0900 - 1400

You must give reasons for your answers.

Question 1.

(a) **(3p)** Find the general solution of the differential equation $y' + 3y = 6$.

(b) **(3p)** Determine all values of t such that the vectors are linearly independent:

$$\mathbf{v}_1 = \begin{pmatrix} t \\ 2 \\ 3 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 6 \\ t \\ 9+t \end{pmatrix}$$

(c) **(3p)** Determine whether the function $f(x, y, z) = x^4 + y^4 + z^4 + z^2$ is convex.

(d) **(3p)** Determine the definiteness of a symmetric 3×3 matrix A with $\det(A) = -6$ and $\mathrm{tr}(A) = 4$.

Question 2.

We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A.

(b) **(6p)** Find a base of the nullspace of A.

(c) **(6p)** Determine the eigenvalue λ such that $\mathbf{v}$ is in the eigenspace E_λ.

(d) **(6p)** Find a base of the space U of vectors that are orthogonal to all vectors in $\mathrm{Null}(A)$.

Question 3.

Let q be the quadratic form given by $q(x, y, z) = 3x^2 + 4xy + 2xz + 4y^2 + 2yz + z^2$, and let p be the function given by $p(x, y, z) = u \cdot e^u$ with $u = u(x, y, z) = 1 - q(x, y, z)$.

(a) **(6p)** Determine the definiteness of the quadratic form q.

(b) **(6p)** Solve the unconstrained problem $\max / \min\ p(x, y, z)$.

Consider the Lagrange problem $\max / \min f(x, y, z) = x + y + z$ when $q(x, y, z) = 4$, where q is the quadratic form given above.

(c) **(6p)** Assume that $(x^*, y^*, z^*; \lambda^*)$ is an admissible point that satisfies the first order conditions. Determine which of the following statements are true, and give reasons for your answers:
(A) If $\lambda^* > 0$, then (x^*, y^*, z^*) is a minimum point
(B) If $\lambda^* > 0$, then (x^*, y^*, z^*) is a maximum point
(C) If $\lambda^* < 0$, then (x^*, y^*, z^*) is a minimum point
(D) If $\lambda^* < 0$, then (x^*, y^*, z^*) is a maximum point

(d) **(6p)** Solve the Lagrange problem.

Question 4.

(a) **(6p)** Solve the differential equation: $4y'' + 4y' - 3y = 8 + 8t - 3t^2$

(b) **(6p)** Solve the system of difference equations: $u_{t+1} = 0.7u_t + 0.8v_t,\ v_{t+1} = 0.4u_t + 0.3v_t$

(c) **(6p)** Solve the initial value problem: $2ty^2 - 4y + (2t^2y - 4t)y' = 0,\ y(1) = 5$

(d) **(6p)** Solve the differential equation: $t^2y'' + 4ty' + 2y = 6$

Hint: To find the homogeneous solution in (d), determine all values of r such that $y = t^r$ is a solution of the homogeneous differential equation.

Exam	Final exam in GRA 6035 Mathematics
Date	November 30th 2022 at 1400 - 1700

You must give reasons for your answers.

Question 1.
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A.
(b) **(6p)** Find a base of the nullspace of A.
(c) **(6p)** Determine the eigenvalue λ such that $\mathbf{v}$ is in the eigenspace E_λ.
(d) **(6p)** Find a base of the space U of vectors that are orthogonal to all vectors in $\mathrm{Null}(A)$.

Question 2.
Let q be the quadratic form given by $q(x,y,z) = 3x^2 + 4xy + 2xz + 4y^2 + 2yz + z^2$, and let p be the function given by $p(x,y,z) = u \cdot e^u$ with $u = u(x,y,z) = 1 - q(x,y,z)$.

(a) **(6p)** Determine the definiteness of the quadratic form q.
(b) **(6p)** Solve the unconstrained problem $\max / \min\ p(x,y,z)$.

Consider the Lagrange problem $\max / \min f(x,y,z) = x + y + z$ when $q(x,y,z) = 4$, where q is the quadratic form given above.

(c) **(6p)** Assume that $(x^*, y^*, z^*; \lambda^*)$ is an admissible point that satisfies the first order conditions. Determine which of the following statements are true, and give reasons for your answers:
(A) If $\lambda^* > 0$, then (x^*, y^*, z^*) is a minimum point
(B) If $\lambda^* > 0$, then (x^*, y^*, z^*) is a maximum point
(C) If $\lambda^* < 0$, then (x^*, y^*, z^*) is a minimum point
(D) If $\lambda^* < 0$, then (x^*, y^*, z^*) is a maximum point
(d) **(6p)** Solve the Lagrange problem.

Question 3.

(a) **(6p)** Solve the differential equation: $4y'' + 4y' - 3y = 8 + 8t - 3t^2$
(b) **(6p)** Solve the system of difference equations: $u_{t+1} = 0.7u_t + 0.8v_t,\ v_{t+1} = 0.4u_t + 0.3v_t$
(c) **(6p)** Solve the initial value problem: $2ty^2 - 4y + (2t^2y - 4t)y' = 0,\ y(1) = 5$
(d) **(6p)** Solve the differential equation: $t^2y'' + 4ty' + 2y = 6$

Hint: To find the homogeneous solution in (d), determine all values of r such that $y = t^r$ is a solution of the homogeneous differential equation.

② Exam 11/2022

1. $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$

a) rk(A):

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{matrix} -1 \\ \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 \end{pmatrix} -1 \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} -1 \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rk(A) = \underline{\underline{3}}$

b) Null(A): $A\underline{x} = \underline{0}$ $\quad (A \mid \underline{0}) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

(3) $-z + w = 0 \quad \underline{z = w}$

(2) $-y + z + w = 0 \quad y = z + w = \underline{2w}$ $\qquad$ w free

(1) $x + y + z = 0 \quad x = -y - z = -2w - w = \underline{-3w}$

Null(A):

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3w \\ 2w \\ w \\ w \end{pmatrix} = w \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \\ 1 \end{pmatrix} \quad (w \text{ free}) \Rightarrow \underline{\text{Base}}: \underline{w} = \underline{\underline{\begin{pmatrix} -3 \\ 2 \\ 1 \\ 1 \end{pmatrix}}}$$

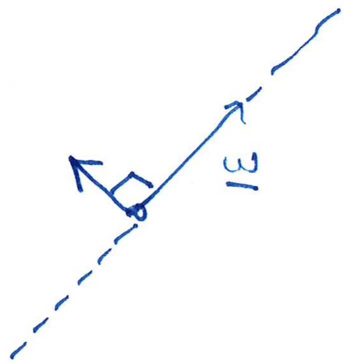
c) <u>Determine λ</u> s.t. $\underline{v}$ is in E_λ : $\underline{v}$ is an eigenvector with eigenvalue λ

$$A\underline{v} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 2 \cdot \underline{v} \qquad A\underline{v} = \lambda \underline{v}$$

$$\Rightarrow \underline{\underline{\lambda = 2}}$$

d) <u>Find a base of</u> U = vector space of vectors orthogonal to Null(A)

$$\text{Null}(A) = \text{span}(\underline{w}), \quad \underline{w} = \begin{pmatrix} -3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$



$\underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in U: $\underline{x} \cdot \underline{w} = 0$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$-3x + 2y + z + w = 0 \quad \leftarrow \text{homogeneous lin. system}$$

$w = 3x - 2y - z$, x, y, z free :

$\underline{x}$ in U:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 3x - 2y - z \end{pmatrix}$$

$$= x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

<u>Base of U:</u>

$$\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}}$$

2. $q(x,y,z) = 3x^2 + 4xy + 2xz + 4y^2 + 2yz + z^2$

$p(x,y,z) = u \cdot e^u$, where $u = 1 - q(x,y,z)$

a) A: symmetric matrix of q

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$D_1 = 3 > 0$

$D_2 = 8 > 0$

$D_3 = +1 \cdot (2-4) - 1 \cdot (3-2) + 1 \cdot 8$

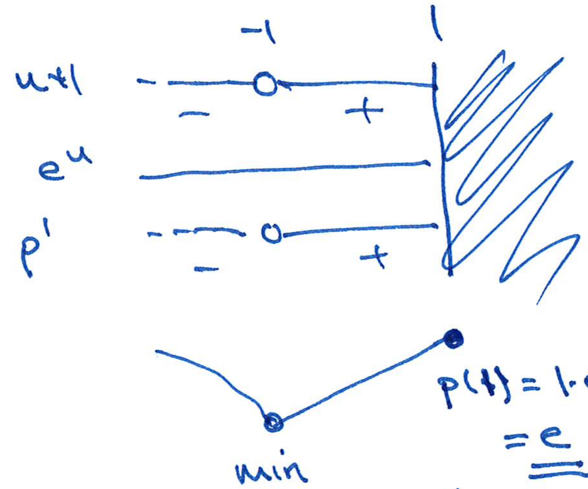
$= -2 - 1 + 8 = \underline{5} > 0$

q $\underline{\underline{\text{pos. defn}}}$.

b) max/min $p(x,y,z) = u \cdot e^u$, where $u = 1 - q(x,y,z)$

$p(u) = ue^u$, $u \leq 1$

$p'(u) = 1 \cdot e^u + u \cdot e^u = (u+1)e^u$



$p(1) = 1 \cdot e^1 = \underline{\underline{e}}$ max

$p(-1) = -1 \cdot e^{-1} = \underline{\underline{-1/e}}$

$\lim_{u \to -\infty} ue^u = 0$

$u(x,y,z) = 1 - q(x,y,z)$

q pos. defn. $\Rightarrow$ $q(x,y,z) \geq 0$

$V(q) = [0, \to)$

$\Rightarrow$ $u(x,y,z) \leq 1$

$\underline{V(u) = (\leftarrow, 1]}$

Concl: max $p_{max} = \underline{\underline{e}}$ at $u = 1$

min $p_{min} = \underline{\underline{-1/e}}$ at $u = -1$

max/min $f = x+y+z$ when $g(x,y,z) = 4$

$= B\underline{x}$ $\qquad$ $\underline{x}^T A \underline{x}$

$B = (1 \ 1 \ 1)$ $\quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$L = B\underline{x} - \lambda(\underline{x}^T A \underline{x} - 4)$

FOC: $L'(\underline{x}) = \boxed{B^T - \lambda(2A\underline{x}) = \underline{0}}$ Lagrange conditions

C: $\boxed{\underline{x}^T A \underline{x} = 4}$

c) If $(x^*, y^*, z^*; \lambda^*)$ satisfies FOC + C, then the SOC can be written:

$h = L(x,y,z;\lambda^*)$ convex $\Rightarrow$ (x^*, y^*, z^*) is min

— " — concave — " — is max

$H(h) = -2\lambda^* A$ $\qquad$ $h = B\underline{x} - \lambda^*(\underline{x}^T A \underline{x} - 4)$

$h'(\underline{x}) = B^T - \lambda^*(2A\underline{x})$

$H(h) = -\lambda^* \cdot 2A = \underline{-2\lambda^* \cdot A}$

A pos. defn $\Rightarrow$ $H(h)$ neg. defn. if $\lambda^* > 0$

h concave $\Rightarrow$ max

(A) False

(B) True

$H(h)$ pos. defn. if $\lambda^* < 0$

h convex $\Rightarrow$ min

(C) True

(D) False

Eivind Eriksen (eivind.eriksen@bi.no), Office B3y-085
https://www.dr-eriksen.no/teaching/GRA6035/
https://www.dr-eriksen.no/teaching/ELE3781/
BI Norwegian Business School
MSc in Business/Finance
BSc in Data Science for Business

d) FOC: $B^T - \lambda(2A\underline{x}) = \underline{0}$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} - \lambda\cdot\left(2A\cdot\underline{x}\right) = \underline{0}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = 2\lambda A\underline{x}$$

$$\begin{pmatrix}3&2&1\\2&4&1\\1&1&1\end{pmatrix}\underline{x} = A\underline{x} = \frac{1}{2\lambda}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\lambda\\1/2\lambda\\1/2\lambda\end{pmatrix} = \begin{pmatrix}t\\t\\t\end{pmatrix}$$

$\lambda \neq 0$
($\lambda = 0$ impossible)
$t = \frac{1}{2\lambda}$

$$\left(\begin{array}{ccc|c}3&2&1&t\\2&4&1&t\\1&1&1&t\end{array}\right) \to \left(\begin{array}{ccc|c}1&1&1&t\\2&4&1&t\\3&2&1&t\end{array}\right)\begin{matrix}\\-2\\-3\end{matrix} \to \left(\begin{array}{ccc|c}1&1&1&t\\0&2&-1&-t\\0&-1&-2&-2t\end{array}\right)$$

$$\to \left(\begin{array}{ccc|c}1&1&1&t\\0&-1&-2&-2t\\0&2&-1&-t\end{array}\right)\begin{matrix}\\\\+2\end{matrix} \to \left(\begin{array}{ccc|c}1&1&1&t\\0&-1&-2&-2t\\0&0&-5&-5t\end{array}\right)$$

(3) $-5z = -5t \Rightarrow \underline{z = t}$
(2) $-y - 2t = -2t \Rightarrow \underline{y = 0}$
(1) $x + 0 + t = t \Rightarrow \underline{x = 0}$

FOC: $(x,y,z) = (0,0,t)$, $t = \frac{1}{2\lambda}$

C: $q(x,y,z) = 4$

$3x^2 + 4xy + 2xz + 4y^2 + 2yz + z^2 = 4$

$t^2 = 4$

$t = \pm 2$

Alt: $(0\ 0\ t)\begin{pmatrix}3&2&1\\2&4&1\\1&1&1\end{pmatrix}\begin{pmatrix}0\\0\\t\end{pmatrix} = t^2 = 4$

$t = \pm 2$

Candidate pts: $(x,y,z;\lambda) = (0,0,2;1/4)$, $\leftarrow$ max } from

(FOC + C) $(0,0,-2;-1/4)$ $\leftarrow$ min } (e)

$t = \frac{1}{2\lambda}$ $2\lambda t = 1$

$\lambda = \frac{1}{2t}$

$\Downarrow$

$f_{max} = f(0,0,2) = \underline{\underline{2}}$ at $(0,0,2)$ $\lambda = 1/4$

$f_{min} = f(0,0,-2) = \underline{\underline{-2}}$ at $(0,0,-2)$ $\lambda = -1/4$

3. a) $4y'' + 4y' - 3y = 8 + 8t - 3t^2$ sec. order lin

$y = y_h + y_p = \underline{\underline{C_1 e^{\frac{1}{2}t} + C_2 e^{-\frac{3}{2}t} + t^2}}$

y_h: $4y'' + 4y' - 3y = 0$ $y_h = \underline{C_1 \cdot e^{\frac{1}{2}t} + C_2 \cdot e^{-\frac{3}{2}t}}$

$4r^2 + 4r - 3 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4}$

$= \frac{-4 \pm \sqrt{64}}{8} = 1/2, -3/2$

y_p: $y = At^2 + Bt + C$

$y' = 2At + B$

$y'' = 2A$

$4 \cdot (2A) + 4(2At + B) - 3(At^2 + Bt + C) = 8 + 8t - 3t^2$

$(-3A)t^2 + (8A - 3B)t + (8A + 4B - 3C)$

$\quad = -3 \qquad\qquad = 8 \qquad\qquad = 8$

$\underline{A = 1}$ $\quad 8 \cdot 1 - 3B = 8$, $\underline{B = 0}$ $\quad 8 \cdot 1 + \cancel{4 \cdot 0} - 3C = 8$, $\underline{C = 0}$

$\Downarrow$

$y_p = \underline{t^2}$

b) $u_{t+1} = 0.7u_t + 0.8v_t$
$v_{t+1} = 0.4u_t + 0.3v_t$

$$\begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.8 \\ 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

$$\underline{y}_{t+1} = A \cdot \underline{y}_t$$

General Solution:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = c_1 \cdot \underline{v_1} \cdot \lambda_1^t + c_2 \cdot \underline{v_2} \cdot \lambda_2^t$$

$$= c_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot 1.1^t + c_2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot (-0.1)^t$$

Eigenvalues:

$$\lambda^2 - 1 \cdot \lambda + 0.11 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1^2 - 4(-0.11)}}{2}$$

$$= \frac{1 \pm 1.2}{2} = \underline{1.1}, \underline{-0.1}$$

$\lambda = 1.1$:

$$\begin{pmatrix} -0.4 & 0.8 \\ \cancel{0.4} & \cancel{-0.8} \end{pmatrix}$$

$\Rightarrow \underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$-0.4x + 0.8y = 0$$

$$x = \frac{0.8y}{0.4} = 2y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda = -0.1$:

$$\begin{pmatrix} 0.8 & 0.8 \\ \cancel{0.4} & \cancel{0.4} \end{pmatrix}$$

$$0.8x + 0.8y = 0$$

$$y = -x$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

c) $\underbrace{(2ty^2 - 4y)}_{p} + \underbrace{(2t^2y - 4t)}_{q}y' = 0$, $\underbrace{y(1) = 5}_{\text{initial condition}}$

(1) $h'_t = 2ty^2 - 4y$

(2) $h'_y = 2t^2y - 4t$

General solution:

$h(t,y) = C$

(1) $h = t^2 \cdot y^2 - 4yt + C(y)$

(2) $h'_y = (t^2y^2 - 4yt + C(y))'_y = 2yt^2 - 4t + C'(y)$

$= 2t^2y - 4t$

ok if $C'(y) = 0$

$\Rightarrow$ exact with $h = t^2y^2 - 4yt = C$

$y(1)=5$: $\quad 1^2 \cdot 5^2 - 4 \cdot 1 \cdot 5 = C$

$(t=1, y=5)$ $\quad C = 25 - 20 = 5$

$t^2y^2 - 4ty = 5$

$t^2y^2 - 4ty - 5 = 0$

$$y = \frac{4t \pm \sqrt{(-4t)^2 - 4 \cdot t^2 \cdot (-5)}}{2t^2} = \frac{4t \pm \sqrt{36t^2}}{2t^2}$$

$$= \frac{4t \pm 6t}{2t^2} \Rightarrow y = \frac{10t}{2t^2} = \frac{5}{t} \quad \text{or} \quad \cancel{y = \frac{-2t}{2t^2} = -\frac{1}{t}}$$

$y(1)=5$

Concl: $y = 5/t$

d) $t^2 y'' + 4ty' + 2y = 6$ Second order linear

$$y = y_h + y_p = \underline{\underline{3 + C_1 \cdot \frac{1}{t^2} + C_2 \cdot \frac{1}{t}}}$$

$\underline{y_p:}$

$$\left.\begin{aligned} y &= A \\ y' &= 0 \\ y'' &= 0 \end{aligned}\right\} \quad \begin{aligned} 0 + 0 + 2A &= 6 \\ A &= 3 \\ y_p &= \underline{3} \end{aligned}$$

$\underline{y_h:}$ $t^2 y'' + 4ty' + 2y = 0$

$$\left.\begin{aligned} y &= t^r \\ y' &= r \cdot t^{r-1} \\ y'' &= r \cdot (r-1) t^{r-2} \end{aligned}\right\} \quad \begin{aligned} &t^2 \cdot r(r-1) t^{r-2} + 4t r \cdot t^{r-1} + 2t^r = 0 \\ &r(r-1) \cdot t^r + 4r \cdot t^r + 2 \cdot t^r = 0 \\ &t^r \cdot (\underbrace{r(r-1) + 4r + 2}) = 0 \end{aligned}$$

$$r(r-1) + 4r + 2 = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$\underline{r = -2}, \quad \underline{r = -1}$$

$$y_h = C_1 \cdot t^{-2} + C_2 \cdot t^{-1}$$

$$= \underline{C_1 \cdot \frac{1}{t^2} + C_2 \cdot \frac{1}{t}}$$

From Final exam 12/2021 Q3

Kuhn-Tucker
Envelope Thm.

max $f = x+y+z+w$ when $3x^2+2xy+8xz-2xw+y^2+4yz+2yw+7z^2+4w^2 \le 18$

$= B\underline{x}$

$\underline{x}^T A \underline{x} \le 18$

$B = (1 \quad 1 \quad 1 \quad 1)$

KT std form

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$$

$L = B\underline{x} - \lambda(\underline{x}^T A \underline{x} - 18)$

FOC: $L'(\underline{x}) = B^T - \lambda(2A\underline{x}) = \underline{0}$

C: $\underline{x}^T A \underline{x} \le 18$

CSC: $\lambda \ge 0$ and $\lambda = 0$ if $\underline{x}^T A \underline{x} < 18$

$\Leftrightarrow$ $\boxed{\underline{x}^T A \underline{x} = 18 \text{ and } \lambda \ge 0}$ or $\boxed{\cancel{\underline{x}^T A \underline{x} < 18 \text{ and } \lambda = 0}}$

Find candidate pts:

FOC: $B^T - 2\lambda \cdot A\underline{x} = \underline{0}$

$B^T = 2\lambda A\underline{x}$

$$\begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}\underline{x} = A\underline{x} = \frac{1}{2\lambda} \cdot B^T = \frac{1}{2\lambda}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix} \qquad t = \frac{1}{2\lambda}$$

$\lambda = 0$: $B^T = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ impossible $\Rightarrow$ $\lambda > 0$ and $\underline{x}^T A \underline{x} = 18$

Gauss

FOC → . . . → $\quad$ $x = t$, $y = 2t$, $z = -t$, $w = 0$

(see sol's of 3a)

C, binding:

$g(t, 2t, -t, 0) = 18$

$3t^2 + 2t(2t) + 8t(-t) + (2t)^2$

$+4(2t)(-t) + 7(-t)^2 = 18$

$2t^2 = 18$

$t^2 = 9$

$t^2 = 9$

$t = \pm 3 \qquad t = \frac{1}{2\lambda}$

$t = 3 \qquad \lambda = \frac{1}{2 \cdot 3} = \frac{1}{6}$

$$\boxed{\text{max: } (x, y, z, w; \lambda) = (3, 6, -3, 0; 1/6) \quad f_{max} = 6}$$

Envelope thm.:

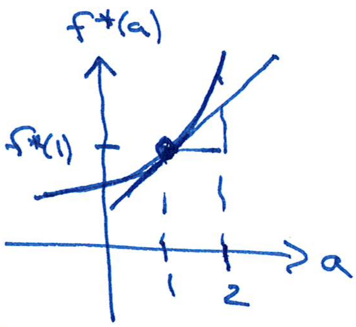
max $f = 2x + y + z + w$ when $g(x, y, z, w) \le 18$

→ max $f = ax + y + z + w$ $\quad$ " $\quad$ — " —

$\mathcal{L} = ax + y + z + w - \lambda(g(\underline{x}) - 18)$

$f^*(1) =$ max value I found before $= 6$

$f^*(2) =$ new max value $(a = 2)$



$$f^*(2) \simeq f^*(1) + \Delta a \cdot \frac{df^*(a)}{da} = f^*(1) + 3 = 6 + 3 = 9$$

$f^*(1) = 6$, $\Delta a = 1$, $\frac{df^*(a)}{da} = \mathcal{L}'_a(\underline{x}^*(a); \lambda^*(a))$ Env. Thm

In this case:

$$\mathcal{L}'_a = x \Rightarrow \frac{df^*(a)}{da} = x^*(a) = 3$$