Plan

Review:
- Solve linear systems using Gaussian elimination
- pivot positions
- rank of a matrix

Ex:

$$x+y+z+w=5$$
$$x-y+3z-w=5$$
$$x+3y-z+3w=5$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & -1 & 3 & -1 & 5 \\ 1 & 3 & -1 & 3 & 5 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x+y+z+w=5$
$\underline{-2y}+2z-2w=0$
z, w free

$$(x,y,z,w) = (-2z+5,\ z-w,\ z,\ w)$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z+5 \\ z-w \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z \\ z \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -w \\ 0 \\ w \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= z \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \underbrace{z \cdot \underline{w}_1 + w \cdot \underline{w}_2}_{\text{span}(\underline{w}_1, \underline{w}_2)} + \underline{w}_0$$

$$x+y+z+w=0$$
$$x-y+3z-w=0$$
$$x+3y-z+3w=0$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & -1 & 0 \\ 1 & 3 & -1 & 3 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z \\ z-w \\ z \\ w \end{pmatrix}$$

$$\underline{x} = z \cdot \underline{w}_1 + w \cdot \underline{w}_2$$

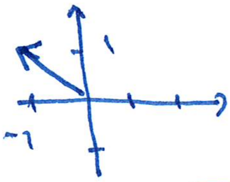
Solutions: $\text{span}(\underline{w}_1, \underline{w}_2)$

1 Introduction to vectors

Defn: A column vector is a matrix with one column.

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

Ex: $\underline{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$$\|\underline{v}\| = \sqrt{(-1)^2 + 1^2} = \underline{\sqrt{2}}$$

$$\underline{v} = (-1,1)$$

Alternative descr:

A vector is a quantity with a magnitude and a direction

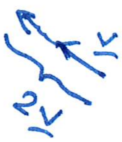
= arrows

Ex:

Scalar mult:

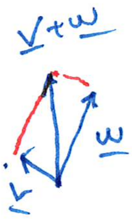
$r \cdot \underline{v}$

$$2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

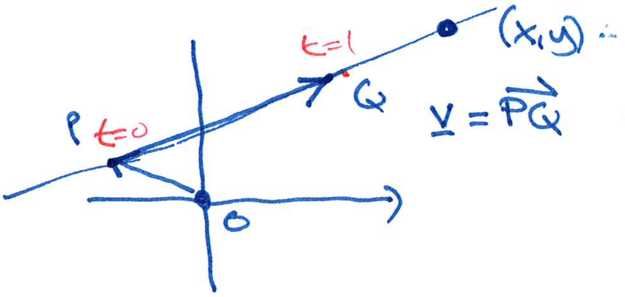


Vector addition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$



Ex: Parametric representation of a line through two points P, Q.



$\underline{v} = \vec{PQ}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{OP} + t \cdot \vec{PQ}.$$

Ex: $P = (-1,1) \quad Q = (3,3)$

$$\underline{v} = (4,2) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 + 4t \\ 1 + 2t \end{pmatrix}$$

② Vector spaces and the span of vectors

$\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ m-vectors

Defn: An expression $c_1 \cdot \underline{v_1} + c_2 \underline{v_2} + \dots + c_n \cdot \underline{v_n}$ (where $c_1, c_2, \dots, c_n$ are any numbers) is called a *linear combination* of $\underline{v_1}, \dots, \underline{v_n}$.

Ex:

$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$2\underline{v_1} + 3\underline{v_2}$

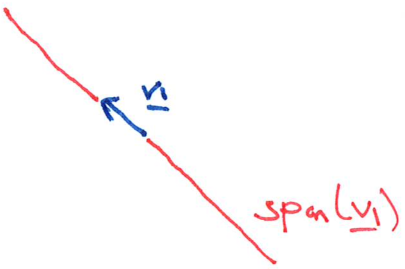
$= 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

Defn: $\text{span}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n})$ = the set of all linear comb. of $\underline{v_1}, \dots, \underline{v_n}$.

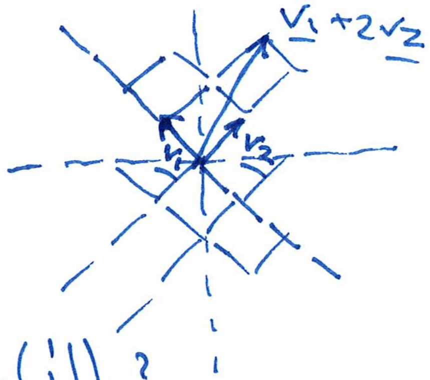
span of $\underline{v_1}, \dots, \underline{v_n}$

$= \{ c_1 \cdot \underline{v_1} + c_2 \underline{v_2} + \dots + c_n \underline{v_n} : c_1, \dots, c_n \text{ are numbers} \}$

Ex: $\text{span}(\underline{v_1}) = \{ c \cdot \underline{v_1} \mid c \text{ number} \}$



Ex: $\text{span}(\underline{v_1}, \underline{v_2}) = \{ c_1 \underline{v_1} + c_2 \underline{v_2} \mid c_i \}$



Ex Is $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ in $\text{span}\left(\underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{= \underline{v_1}}, \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{= \underline{v_2}} \right)$?

$$x_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\begin{aligned} -x_1 + x_2 &= 3 \\ x_1 + x_2 &= 3 \end{aligned} \implies \begin{aligned} x_1 &= 0 \\ x_2 &= 3 \end{aligned}$$

$0 \cdot \underline{v_1} + 3 \cdot \underline{v_2} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

Yes!

Remember:

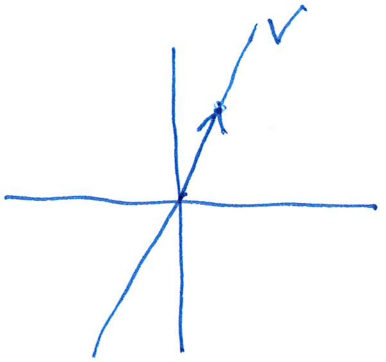
If $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ are m-vectors, then $\text{span}(\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n})$ is a (infinite) subspace of $\mathbb{R}^m$.

Defn: A subset of $\mathbb{R}^m$ is a vector space V if the following conditions hold:

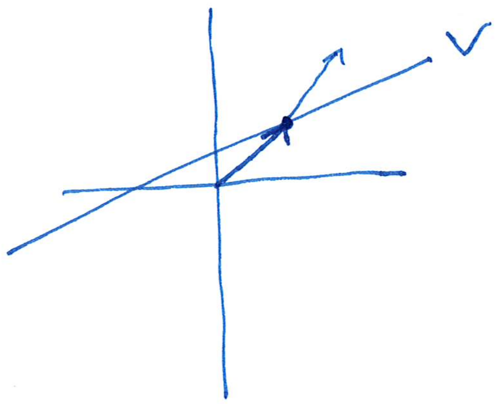
i) If $\underline{v}$ is in V, then $r \cdot \underline{v}$ is in V (for any no. r)

ii) If $\underline{v}, \underline{w}$ in V, then $\underline{v} + \underline{w}$ is in V

Ex:



V is a vector space



V is not a vector space

Result: A subset V in $\mathbb{R}^m$ is a vector space $\iff$ $V = \text{span}(\underline{v_1}, \ldots, \underline{v_r})$ for some vectors $\underline{v_1}, \ldots, \underline{v_r}$.

Remark: The solutions of a homogeneous linear system $A \cdot \underline{x} = \underline{0}$ is a vector space called $\underline{\text{Null}(A)}$

In the non-homogeneous case $A\underline{x} = \underline{b}$ $(\underline{b} \neq \underline{0})$ then the solutions are not a vector space.

In fact, any homogeneous linear system $A \cdot \underline{x} = \underline{0}$ has solution

$$\underline{x} = t_1 \cdot \underline{w_1} + t_2 \underline{w_2} + \dots + t_r \cdot \underline{w_r}$$

where $t_1, t_2, \dots, t_r$ are the free variables, r is the no. of degrees of freedom, and $\underline{w_1}, \underline{w_2}, \dots, \underline{w_r}$ are vectors you have to compute. This means that

$$\boxed{\underset{\text{all solutions of } A\underline{x}=\underline{0}}{\mathrm{Null}(A)} = \mathrm{span}(\underline{w_1}, \underline{w_2}, \dots, \underline{w_r})}$$

$r = \#$ degrees of freedom. $= n - \mathrm{rk}(A)$

③ Linear independence

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$V = \mathrm{span}(\underline{v_1}, \underline{v_2}, \underline{v_3})$

$\underline{0} = x_1 \underline{v_1} + x_2 \underline{v_2} - 1 \cdot \underline{v_3}$

$\left.\begin{array}{l} \underline{v_3} = x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} \\ \underline{v_2} = x_1 \underline{v_1} + x_3 \cdot \underline{v_3} \\ \underline{v_1} = x_2 \underline{v_2} + x_3 \underline{v_3} \end{array}\right\}$ $x_1 \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = \underline{0}$

Defn: The vectors $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ are linearly dependent if one of the vectors are a linear combination of the others. Otherwise, they are linearly independent.

Result: $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ are linearly independent $\iff$ $x_1 \underline{v_1} + x_2 \underline{v_2} + \dots + x_n \cdot \underline{v_n} = \underline{0}$ has only the trivial solution $\underline{x} = \underline{0}$

homogeneous linear system $A \cdot \underline{x} = \underline{0}$

Ex: $x_1\begin{pmatrix}1\\2\\1\end{pmatrix}+x_2\cdot\begin{pmatrix}1\\3\\2\end{pmatrix}+x_3\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$x_1+x_2+x_3=0$$
$$2x_1+3x_2-x_3=0$$
$$x_1+2x_2-2x_3=0$$

$$\begin{array}{ccc}\left(\begin{array}{ccc|c}1&1&1&0\\2&3&-1&0\\1&2&-2&0\end{array}\right)&\to&\left(\begin{array}{ccc|c}1&1&1&0\\0&1&-3&0\\0&1&-3&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&0\\0&1&-3&0\\0&0&0&0\end{array}\right)\\ \uparrow\ \uparrow\ \uparrow&&\\ \underline{v_1}\ \underline{v_2}\ \underline{v_3}&&\end{array}$$

one free variable x_3

$\Rightarrow$ $\underline{v_1},\underline{v_2},\underline{v_3}$ are linearly dependent

$\underline{v_3}=4\underline{v_1}-3\underline{v_2}$

$$x_1+x_2+x_3=0$$
$$x_2-3x_3=0$$

$x_2=3x_3$ $\quad x_1=-x_2-x_3=-3x_3-x_3=-4x_3$

$$\underline{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-4x_3\\3x_3\\x_3\end{pmatrix}=x_3\cdot\begin{pmatrix}-4\\3\\1\end{pmatrix}=t\cdot\underline{w},\quad \underline{w}=\begin{pmatrix}-4\\3\\1\end{pmatrix}$$

$$-4\underline{v_1}+3\underline{v_2}+\underline{v_3}=\underline{0}$$
$$\underline{v_3}=4\underline{v_1}-3\underline{v_2}$$

Interpretation:

$V=\text{span}(\underline{v_1},\underline{v_2},\underline{v_3})=\text{span}(\underline{v_1},\underline{v_2})$

$c_1\underline{v_1}+c_2\underline{v_2}+c_3(4\underline{v_1}-3\underline{v_2})$

$(c_1+4c_3)\underline{v_1}+(c_2-3c_3)\underline{v_2}$

Base of V:

$\{\underline{v_1},\underline{v_2}\}$

Dimension of V: 2

Dimension

Let V be a vector space, i.e $V = \text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n)$.

Defn: A base of V is a set of vectors such that

i) the vectors span V

ii) the vectors are linearly independent

The dimension of V is the number of vectors in a base of V, and is written $\dim(V)$.

Some important vector spaces:

i) $\underset{m\text{-vectors}}{\underline{v}_1, \ldots, \underline{v}_n}$: $V = \text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n) = \text{Col}(A)$

$\updownarrow$

$A = (\underline{v}_1 \mid \underline{v}_2 \mid \ldots \mid \underline{v}_n)$

$m \times n$-matrix

Col(A) = column space of A = span of the col's of A.

Result: $\dim V = \dim \text{Col}(A) = \text{rk}(A)$

Base: vectors with pivot positions

ii) $\text{Null}(A)$ = all solutions of $A \cdot \underline{x} = \underline{0}$. $= t_1 \cdot \underline{w}_1 + t_2 \cdot \underline{w}_2 + \ldots + t_r \cdot \underline{w}_r$

A: $m \times n$-matrix

where r is the no. of free var's, $t_1, \ldots, t_r$ are the free var's, and $\underline{w}_1, \ldots, \underline{w}_r$ are vectors

Result: $\dim \text{Null}(A) = \#\text{free variables} = n - \text{rk}(A)$

Base: $\{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_r\}$

Ex: $A = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & 1 & 4 & 1 \\ 3 & 2 & 3 & 4 \end{pmatrix}$ $\quad \underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ $\underline{v_4} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

3x4 matrix

i) $V = \text{col}(A) = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4})$

$\dim \text{Col}(A) = \text{rk} A = \underline{\underline{2}}$

Base: $\{\underline{v_1}, \underline{v_2}\}$

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & 1 & 4 & 1 \\ 3 & 2 & 3 & 4 \end{pmatrix} \begin{matrix} -2 \\ -3 \end{matrix} \to \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 6 & -5 \\ 0 & -1 & 6 & -5 \end{pmatrix} \begin{matrix} \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 6 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} + x_4 \underline{v_4} = \underline{0}$

$x_1 + x_2 - x_3 + 3x_4 = 0$

$-x_2 + 6x_3 - 5x_4 = 0$

$x_2 = \underline{6x_3 - 5x_4}$

$x_1 = -6x_3 + 5x_4 + x_3 - 3x_4$

$= \underline{-5x_3 + 2x_4}$

$$\underline{x} = \begin{pmatrix} -5x_3 + 2x_4 \\ 6x_3 - 5x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -5 \\ 6 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 2 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

$-5\underline{v_1} + 6\underline{v_2} + \underline{v_3} = \underline{0}$

$2\underline{v_1} - 5\underline{v_2} + \underline{v_4} = \underline{0}$

$\boxed{\begin{aligned} \underline{v_3} &= 5\underline{v_1} - 6\underline{v_2} \\ \underline{v_4} &= -2\underline{v_1} + 5\underline{v_2} \end{aligned}}$

$\boxed{\text{Col}(A) = \text{span}(\underline{v_1}, \underline{v_2})}$

ii) $W = \text{Null}(A)$: Solutions of $A\underline{x} = \underline{0}$

$W = \text{span}(\underline{w_1}, \underline{w_2})$ $\quad \underline{w_1} = \begin{pmatrix} -5 \\ 6 \\ 1 \\ 0 \end{pmatrix}$ $\underline{w_2} = \begin{pmatrix} 2 \\ -5 \\ 0 \\ 1 \end{pmatrix}$

$\dim \text{Null}(A) = n - \text{rk}(A) = 2$

Base: $\{\underline{w_1}, \underline{w_2}\}$

4 Inner products and projections

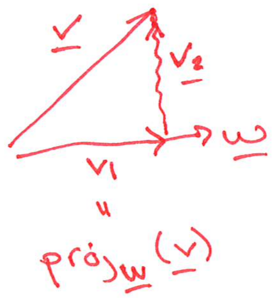
$$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} : \quad \underline{v} \cdot \underline{w} = v_1 w_1 + \dots + v_m \cdot w_m$$

Ex: $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 1 \cdot 2 + 3 \cdot (-1) + 1 \cdot 0 = \underline{-1}$

Note: $\underline{v} \cdot \underline{w} = 0 \Leftrightarrow \underline{v}$ and $\underline{w}$ are orthogonal (90° degrees)

$\underline{v} \cdot \underline{v} = \|\underline{v}\|^2 \qquad \|\underline{v}\| = $ length of $\underline{v}$

$\underline{v} \cdot (a\underline{w}_1 + b\underline{w}_2) = a(\underline{v} \cdot \underline{w}_1) + b \cdot (\underline{v} \cdot \underline{w}_2)$



Projection of $\underline{v}$ onto $\underline{w}$ (orthogonal projection)

$\underline{v} = \underline{v}_1 + \underline{v}_2$ where $\underline{v}_1$ is along $\underline{w}$

$\underline{v}_2$ is orthogonal to $\underline{w}$

$$\text{proj}_{\underline{w}}(\underline{v}) = \underline{v}_1$$

Formula: $\text{proj}_{\underline{w}}(\underline{v}) = \left(\dfrac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \right) \cdot \underline{w}$