Plenary Session I:

Monday Sep 18 at 16-19
Send me suggestions of problems by email.

Review: A vector space V is a subspace of $\mathbb{R}^m$ that can be written $V = \text{span}(\underline{v_1}, \dots, v_n)$ for some vectors $\underline{v_1}, \dots, \underline{v_n}$

all lin. comb. $c_1\underline{v_1} + c_2\underline{v_2} + \dots + c_n\underline{v_n}$

A base of V is a minimal (linearly independent) set of vectors that spans V

The dimension of V is the no. of vectors in a base of V

dim $V = 0$: V = a point

dim $V = 1$: V = a line

dim $V = 2$: V = a plane

Key method:

$\underline{v_1}, \dots, \underline{v_n}$ m-vectors

$\Updownarrow$

$A = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$ $m \times n$-matrix

Equation:

$(*)$ $x_1\underline{v_1} + \dots + x_n\underline{v_n} = \underline{0}$

$\Updownarrow$

$A \cdot \underline{x} = \underline{0}$

Key result:

$\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ linearly independent

$\Updownarrow$

$(*)$ has only the trivial solution $\underline{x} = \underline{0}$

$\Updownarrow$

rk $A = n$

Compute $A \cdot B$ (matrix mult.)
A^{-1} (inverse)

See EBK 1003 Lecture 3, 5

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① Determinants

$A \rightsquigarrow \det(A) = |A|$

$n \times n$-matrix — determinant of A, a number

Methods for computing the determinant:
- Cofactor exp.
- using Gauss

Ex: $A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$ 4x4-matrix

$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{vmatrix} = ?$

Fact: Cofactor expn. through any row or column gives the same result, det(A).

Alt 1: Cofactor expansion

$$\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot C_{11} + 0 \cdot C_{12} + 0 \cdot C_{13} + 4 \cdot C_{14}$$

$$= +1 \cdot \begin{vmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 0 \cdot * + 0 * - 4 \cdot \begin{vmatrix} 0 & 2 & 3 \\ 0 & 3 & 2 \\ 4 & 0 & 0 \end{vmatrix}$$

Signs: $\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$

$$= +1 \left(+1 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\right) - 4 \left(+4 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\right)$$

$$= 1 \cdot (2 \cdot 2 - 3 \cdot 3) - 4 \cdot 4 \, (2 \cdot 2 - 3 \cdot 3)$$

$$= 1 \cdot (-5) - 16 \, (-5) = \underline{\underline{75}}$$

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Alt 2: Using Gaussian elimination

$$\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{vmatrix} \; -4 \; = \begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & -15 \end{vmatrix} \; -3/2 \; =$$

$$2 - 3\cdot 3/2 = 2 - 9/2 = -5/2$$

$$= \begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -5/2 & 0 \\ 0 & 0 & 0 & -15 \end{vmatrix} = 1\cdot 2\cdot(-5/2)(-15) = +75$$

Note: $A \to B$ elementary row operation

$|B| = |A|$ if we add a multiple of one row to another row

$|B| = -|A|$ if we switch two rows

$|B| = c\cdot|A|$ if we multiply a row by $c \neq 0$

Result: If a matrix is upper triangular (in particular, all echelon forms are like this), the determinant is the product of diagonal entries

Note:
$$\begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -5/2 & 0 \\ 0 & 0 & 0 & -15 \end{vmatrix} = +1\cdot\begin{vmatrix} 2 & 3 & 0 \\ 0 & -5/2 & 0 \\ 0 & 0 & -15 \end{vmatrix} = +1\left(+2\begin{vmatrix} -5/2 & 0 \\ 0 & -15 \end{vmatrix}\right)$$

$$= 1\cdot 2\cdot(-5/2)\cdot(-15)$$

Note:

$$\underset{n\times n}{A} \to \cdots \to \underset{\substack{n\times n\\ \text{echelon}\\ \text{form}}}{E} = \underbrace{\begin{pmatrix} e_1 & * & \cdots & * \\ 0 & e_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_n \end{pmatrix}}_{\text{upper triangular}} \qquad |A| = e_1 \cdot e_2 \cdot \ldots \cdot e_n$$

Two cases: ① All $e_i \neq 0 \Rightarrow |A| \neq 0$ "pivot in all col's"

② At least one $e_i = 0 \Rightarrow |A| = 0$ "at least one col. without pivots"

Ex: ① $E = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $|E| = 1 \cdot 3 \cdot 1 = 3 \neq 0$

② $E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $|E| \neq 1 \cdot 0 \cdot 0 = 0$

Result:

$A\underline{x} = \underline{b}$ linear system with A $n\times n$.

Then:

$|A| \neq 0 \Rightarrow$ one solution

$|A| = 0 \Rightarrow$ no solution or inf. many sol's

Result:

$\underset{n\times n}{A}$ **Then:** $|A| \neq 0 \Leftrightarrow rk(A) = n$

$|A| = 0 \Leftrightarrow rk(A) < n$

Theorem: A $n\times n$-matrix

the following are equivalent:

i) $|A| \neq 0$

ii) $rk(A) = n$

iii) A invertible $\left(A^{-1} = \frac{1}{|A|} \cdot adj(A) \right)$

iv) $A\underline{x} = \underline{b}$ has one unique solution for any vector $\underline{b}$

v) The rows of A are linearly independent

vi) The col's of A — " —

Note: i) Some useful formules for the determinant: $\boxed{\begin{aligned} |AB| &= |A| \cdot |B| \\ |A^T| &= |A| \end{aligned}}$

ii) The determinant of A is zero if one of the rows of A is a linear combination of the other rows (since a row of zeros gives that the det. is zero)
(also works with colums)

Ex: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0$

$R(3) = R(1) + R(2)$

Ex: $\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 0$

$C(4) = -C(1)$

② Minors and rank

A $m \times n$ matrix

Defn: An r-minor of A is the determinant of an $r \times r$ submatrix of A.

$M_{I,J}$ = determinant of the submatrix where we include rows $= I$ and cols $= J$

Defn: $M_{I,J}$ is a principal minor if $I = J$.

Ex:

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ & \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 4 \\ 4 & 2 & 1 & 5 \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

3×4 matrix

3-minors:

$$M_{123,123} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= +1 \cdot (-7) - 1(-10) + 1(8)$$

$$= \underline{11} \neq 0$$

$$\Rightarrow \text{rk}\, A = 3$$

(4 3-minors)

2-minors:

$$M_{23,12} = \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = \underline{8}$$

Method: Using **minors** to find the **rank** = rank

Ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 4 \\ 3 & 0 & 4 & 6 \end{pmatrix}$

3x4 matrix

not pivot pos.

Start with max minor = 3-minors :

$M_{12,12} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3 \neq 0$

$$M_{123,123} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 0 & 4 \end{vmatrix} = +3\cdot(4) + 4(-3) = \underline{0}$$

$$M_{123,124} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{vmatrix} = +3(5) + 6(-3) = -3 \neq 0$$

There is a 3-minor $\neq 0 \Rightarrow rk(A) = \underline{\underline{3}}$

Remember: $\underset{n\times n}{A}$ $|A| \neq 0 \Leftrightarrow rk(A) = n$

Result: $\boxed{\text{The rank of } A \text{ is the maximal value of } r \text{ such that at least one } r\text{-minor is non-zero.}}$

Ex: $\underset{3\times 4\text{-matrix}}{A}$

i) One of the 3-minors $\neq 0 \Rightarrow rk\, A = 3$

ii) All 3-minors $= 0 \Rightarrow rk\, A < 3$

a) One 2-minor $\neq 0 \Rightarrow rk\, A = 2$

b) All 2-minors $= 0 \Rightarrow rk\, A < 2$

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Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \end{pmatrix}$

r=2: $M_{12,12} = \begin{vmatrix} 1 & 1 \\ 1 & p \end{vmatrix} = p-1$

$M_{12,13} = \begin{vmatrix} 1 & 1 \\ 1 & p^2 \end{vmatrix} = p^2-1$

$M_{12,23} = \begin{vmatrix} 1 & 1 \\ p & p^2 \end{vmatrix} = p^2-p$

$\text{rk}\, A < 2 \iff$ All 2-minors are zero $\iff$ $p-1=0$, $p^2-1=0$, $p^2-p=0$ $\Rightarrow p=1$

$p=1$: $\text{rk}\, A < 2$ $\quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad \text{rk}\, A = 1$

$p \neq 1$: $\text{rk}\, A = 2$

Conclusion: $\text{rk}\, A = \begin{cases} 1, & p=1 \\ 2, & p \neq 1 \end{cases}$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \\ 1 & q & q^2 \end{pmatrix}$

$|A| = 1(pq^2 - p^2q) - 1(q^2-p^2) + 1\cdot(q-p)$

$= pq(q-p) - (q-p)(q+p) + (q-p)$

$= (q-p)\cdot[pq - (q+p) + 1]$

$\text{rk}\, A < 3 \iff |A| = 0$

$\Updownarrow$

$(q-p)(pq-p-q+1) = 0$

$q=p$ or $pq-p-q+1=0$

$(p-1)(q-1)=0$

$p=1$ or $q=1$

$\operatorname{rk} A < 3$ if $p=q$ or $p=1$ or $q=1$

$\underline{p=1}$: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & q & q^2 \end{pmatrix}$ $\quad q \neq 1$: $M_{13,12} = \begin{vmatrix} 1 & 1 \\ 1 & q \end{vmatrix} = q-1 \neq 0$

$\Rightarrow \operatorname{rk} A = 2$

$q=1$: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\operatorname{rk} A = 1$

$p \neq 1, \underline{q=1}$: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad M_{12,12} = p-1 \Rightarrow \operatorname{rk} A = 2$ if $p \neq 1$

$p \underline{\neq 1, q \neq 1}$: $\quad p=q$: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \\ 1 & p & p^2 \end{pmatrix}$ $\quad M_{12,12} = p-1 \neq 0$

$\operatorname{rk} A = \underline{2}$

$p \neq q$: $|A| \neq 0 \quad \operatorname{rk} A = 3$

$\underline{\text{Conclusion}}$: $\operatorname{rk} A = \begin{cases} 1 & \text{if } p=q=1 \\ 2 & \text{if } p=1 \text{ or } q=1 \text{ or } p=q \text{ but } (p,q) \neq (1,1) \\ 3 & \text{if } p \neq 1, q \neq 1, p \neq q \end{cases}$

3 Applications to linear systems

Ex:
$$x+y+z+w = 5$$
$$2x - y + 3z + 4w = 2$$
$$3x + 4z + 6w = -1$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 4 \\ 3 & 0 & 4 & 6 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$M_{123,123} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 0 & 4 \end{vmatrix} = 0$$

since $R(3) = R(1) + R(2)$

$$M_{123,124} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{vmatrix} = 3(+5) + 6(-3) = -3 \neq 0$$

$$\left.\begin{array}{l} \Rightarrow \text{rk } A = 3 \\ \text{rk}(A|b) = 3 \end{array}\right\} \Rightarrow$$ one degree of freedom (z free), inf. many solutions

(see earlier in the lecture)

Explicit solutions:

z free

$$x + y + w = 5 - z$$
$$2x - y + 4w = 2 - 3z$$
$$3x + 6w = -1 - 4z$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix}$$

invertible since the determinant is $M_{123,124} = -3 \neq 0$

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix}$$

adj. matrix = transpose of cofact. matrix

$$= \frac{1}{-3} \begin{pmatrix} -6 & -6 & +5 \\ 6 & +3 & -2 \\ 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 6 & 6 & -5 \\ 0 & -3 & 2 \\ -3 & -3 & 3 \end{pmatrix} \begin{pmatrix} 5-z \\ 2-3z \\ -1-4z \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 47-4z \\ z-8 \\ -24 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4/3\,t - 4/3 \\ 1/3\,t - 8/3 \\ +t \\ -8 \end{pmatrix}$$ //