Plan

1 Eigenvalues and eigenvectors
2 Diagonalization
3 Application: Powers of matrices

Reminder:
- Plenary Session 1 Monday
- Possible to request problems (incl. Problem Set 4) until Monday

① Eigenvalues and eigenvectors

A $n\times n$ matrix

(*) $A\cdot \underline{v} = \lambda\cdot\underline{v}$ $\begin{cases} \underline{v}: \text{vector} \\ \lambda: \text{number}\end{cases}$

Defn: λ is an eigenvalue of A if (*) has non-trivial solutions $\underline{v} \neq \underline{0}$.

$\underline{v}$ is an eigenvector of A with eigenvalue λ if $\underline{v}\neq\underline{0}$ is a solution of (*).

(*) $A\cdot\underline{v} = \lambda\underline{v}$

$A\underline{v} - \lambda\underline{v} = \underline{0}$

$\boxed{(A-\lambda I)\underline{v} = \underline{0}}$ $\begin{cases}\text{lin. homogeneous} \\ \text{system w/} \\ \text{parameter } \lambda\end{cases}$

$\lambda\cdot I = \begin{pmatrix}\lambda & & 0 \\ & \lambda & \\ 0 & & \lambda\end{pmatrix}$

(a) Computing eigenvalues of A

λ is an eigenvalue of A $\Leftrightarrow$ $|A-\lambda I| = 0$ characteristic equation

Ex: $A = \begin{pmatrix}2 & 1 \\ 1 & 2\end{pmatrix}$

$A\cdot\underline{v} = \begin{pmatrix}2 & 1 \\ 1 & 2\end{pmatrix}\begin{pmatrix}3 \\ -2\end{pmatrix} = \begin{pmatrix}4 \\ -1\end{pmatrix} \neq \lambda\cdot\begin{pmatrix}3 \\ -2\end{pmatrix}$

$\underline{v}$ not eigenvector

$A\cdot\underline{v} = \begin{pmatrix}2 & 1 \\ 1 & 2\end{pmatrix}\begin{pmatrix}1 \\ 1\end{pmatrix} = \begin{pmatrix}3 \\ 3\end{pmatrix} = 3\cdot\begin{pmatrix}1 \\ 1\end{pmatrix} = \lambda\cdot\underline{v}$

$\lambda = 3$ eigenvalue

$\underline{v} = \begin{pmatrix}1 \\ 1\end{pmatrix}$ is an eigenvector with $\lambda = 3$

$A\cdot\underline{v} = \begin{pmatrix}2 & 1 \\ 1 & 2\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \lambda\cdot\begin{pmatrix}x \\ y\end{pmatrix}$

$\boxed{\begin{aligned} 2x+y &= \lambda x \\ x+2y &= \lambda y\end{aligned}}$

$2x - \lambda x + y = 0$
$x + 2y - \lambda y = 0$

$\begin{pmatrix}2-\lambda & 1 \\ 1 & 2-\lambda\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$

homogeneous linear system w/ parameter λ.

$n=2$:

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$(a-\lambda)(d-\lambda)-bc=0$

$\lambda^2 - a\lambda - d\lambda + ad - bc = 0$

$$\boxed{\lambda^2 - \underbrace{(a+d)}_{tr(A)}\lambda + \underbrace{(ad-bc)}_{|A|} = 0}$$

tr(A) = sum of diag. entries

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1^2 = 0$

$4 - 4\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda-3)(\lambda-1)=0$

$\underline{\lambda_1 = 3}$ $\underline{\lambda_2 = 1}$

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\lambda^2+1=0$

$\lambda^2 = -1$ no eigenvalues

$A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $\lambda^2 - 2\lambda + 1 = 0$ $\lambda_1 = 1, \lambda_2 = 1$

$(\lambda-1)^2 = 0$ $\lambda = 1$ with multiplicity 2

$\underset{n\times n}{A}$ Facts:
- the characteristic equation $|A-\lambda I|=0$ is an n'th order polynomial equation
- if A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$\boxed{\begin{aligned} \lambda_1+\lambda_2+\dots+\lambda_n &= tr(A) \\ \lambda_1\cdot\lambda_2\cdot\dots\cdot\lambda_n &= det(A) \end{aligned}}$$

A symm. if $A^T = A$

- if A is symmetric, then it has n eigenvalues (counted with multiplicity)

(b) Computing eigenvectors of A when the eigenvalues are known

λ eigenvalue (given) : Eigenvectors of A with eigenvalue λ = All solutions of $A\underline{v} = \lambda\underline{v}$ $(A-\lambda I)\underline{v} = 0$

|| $= Null(A-\lambda I)$

Eigenspace E_λ

$$\boxed{E_\lambda = Null(A-\lambda I)}$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\qquad \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\underline{\lambda_1 = 3}, \ \underline{\lambda_2 = 1}$

$\underline{\lambda = 3}$: $E_3 = \text{Null}(A - 3I) = \text{Null}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

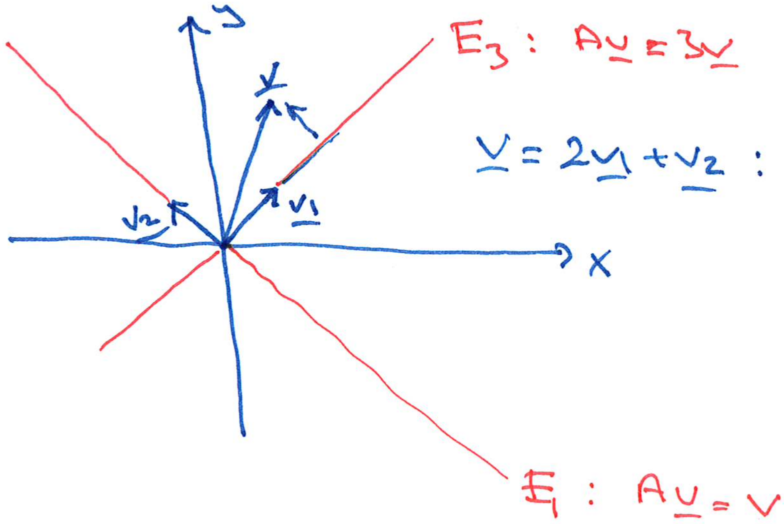
$$\underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_{(A-3I)} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\underline{v}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\underline{0}} : \quad \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad -x + y = 0 \quad \underline{x = y}$, y free

$E_3 = \text{span}(\underline{v_1}), \ \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \underline{v_1}$ is a base of E_3

$\underline{\lambda = 1}$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad x + y = 0$, y free $\qquad x = -y$, y free

$\underline{v} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\qquad E_1 = \text{span}(\underline{v_2})$ $\qquad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$\underline{v} = 2\underline{v_1} + \underline{v_2}$: $A\underline{v} = A(2\underline{v_1} + \underline{v_2})$

$= 2A\underline{v_1} + A\underline{v_2}$

$= 2(3\underline{v_1}) + 1 \cdot \underline{v_2}$

$= 6\underline{v_1} + \underline{v_2}$

Fact: If λ is an eigenvalue of A with multiplicity m, then $1 \leq \dim E_\lambda \leq m$.

$\dim E_\lambda$ = number of free variables in the lin. sys.

$(A - \lambda I)\underline{v} = \underline{0}$

(2) Diagonalization

A $n \times n$ matrix

Defn. A is diagonalizable if there is an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D \quad \leftarrow \text{diagonalization.}$$

Assume:

$A\underline{v_1} = \lambda_1 \underline{v_1}$

$A\underline{v_2} = \lambda_2 \underline{v_2}$

$\vdots$

$A\underline{v_n} = \lambda_n \cdot \underline{v_n}$

and

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ are linearly independent

Then: $P = \left(\underline{v_1} \middle| \underline{v_2} \middle| \cdots \middle| \underline{v_n}\right)$ invertible

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \quad \text{diagonal}$$

$$A \cdot P = A \cdot (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n}) = (A\underline{v_1} | A\underline{v_2} | \ldots)$$
$$= (\lambda_1 \underline{v_1} | \lambda_2 \underline{v_2} | \ldots)$$
$$PD = (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n}) \cdot \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$
$$= \left(\lambda_1 \underline{v_1} \middle| \lambda_2 \underline{v_2} \middle| \ldots \right)$$

$\Downarrow$

$AP = PD \quad | P^{-1} \cdot$

$P^{-1}AP = D$

diagonalization

Theorem:

A $n\times n$ matrix

A is diagonalizable $\iff$ i) A has n eigenvalues (counted with mult.) ii) A has n linearly independent eigenvectors

In that case, ~~the~~ a diagonalization is given by

$$P = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n) \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Eigenvalues: $\lambda_1 = 3,\ \lambda_2 = 1$

Eigenvectors (bases of the eigenspaces)

$\underbrace{\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\lambda=3}$ $\underbrace{\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\lambda=1}$

Is A diagonalizable: Yes.

Diagonalization: $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $|P| = 1 - (-1) = 2 \neq 0$

$D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

Ex: $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}$ **Eigenvalues:**

$$\begin{vmatrix} 3-\lambda & 0 & 2 \\ 0 & 5-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$+ (5-\lambda) \cdot \begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$-(\lambda-5)(\lambda-5)(\lambda-1)$ $\longrightarrow$ $(5-\lambda)\cdot(\lambda^2 - 6\lambda + 5) = 0$

$\lambda = 5$ or $\underline{\lambda = 5 \text{ or } \lambda = 1}$

$1 \le \dim E_5 \le 2$

$\dim E_1 = 1$

$\lambda_1 = \lambda_2 = 5$ (mult. 2) $\lambda_3 = 1$ (mult 1)

$\lambda = 5$: $\begin{pmatrix} -2 & 0 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{2} & \cancel{0} & \cancel{-2} \end{pmatrix}$ $\quad -2x + 2z = 0$, y, z free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

A is diagonalizable $\quad \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad E_5 = \text{span}(\underline{v}_1, \underline{v}_2)$

$\lambda = 1$: $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ \cancel{2} & \cancel{0} & \cancel{2} \end{pmatrix}$ $\quad 2x + 2z = 0$, $4y = 0$, z free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad E_1 = \text{span}(\underline{v}_3)$

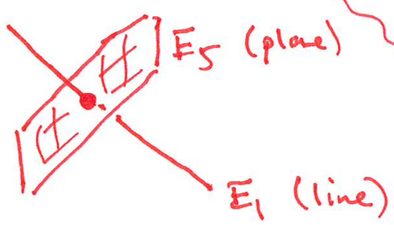
Diagonalization:

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -1 \cdot 2 = -2 \neq 0$$

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$P^{-1}AP = D$

Picture:



Fact: (1) A diagonalizable $\Leftrightarrow$ the sum of dim E_λ for all eigenvalues λ is n

A $n \times n$ matrix

(2) A symmetric $\Rightarrow$ A diagonalizable

(3) A has n distinct eigenvalues $\Rightarrow$ A diagonalizable

Ex:

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 0-\lambda & 1 & -2 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda & 1 & -2 \\ 1 & -\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\underbrace{-\lambda \cdot \left[(-\lambda)(-1-\lambda) - 1\right]} - \underbrace{1 \cdot (-1-\lambda-1)} + \underbrace{(-2)(1+\lambda)} = 0$$

$$-\lambda(\lambda^2+\lambda-1) + \lambda + 2 \quad -2\lambda - 2 = 0$$

$$-\lambda^3 - \lambda^2 \cancel{+\lambda} \cancel{+\lambda} \cancel{+2} \cancel{-2\lambda} \cancel{-2} = 0$$

$$-\lambda^3 - \lambda^2 = 0$$

$$-\lambda^2(\lambda+1) = 0$$

$\lambda_1 = \lambda_2 = 0$ $\quad$ $\lambda_3 = -1$

A is not diagonalizable

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} | & | & \end{pmatrix}$$

Check $\lambda = 0$ (mult=2)

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ \cancel{0} & \cancel{1} & \cancel{-2} \end{pmatrix}$$

z free

$\dim E_0 = 1 < 2 = \text{mult.}$

In more detail:

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Char. equ. (see previous page)

$-\lambda^2(\lambda+1) = 0$

$-\lambda \cdot \lambda \cdot (\lambda+1) = 0$

$\lambda_1 = \lambda_2 = 0$ (mult. 2) $\quad \lambda_3 = -1$ (mult. 1)

Eigenvectors:

$\lambda = 0$:

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

(see prev. page)

$x + z = 0 \qquad x = -z$

$y - 2z = 0 \qquad y = 2z$

z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 2z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \qquad E_0 = \text{span}(\underline{v_1}), \quad v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda = -1$:

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ \cdot(-1) \\ \cdot(-1) \end{matrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{pmatrix} \begin{matrix} \\ \\ -2/3 \end{matrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$x + y - 2z = 0$

$3z = 0$

y free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad E_{-1} = \text{span}(\underline{v_2})$$

$$\underline{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & ? & -1 \\ 2 & ? & 1 \\ 1 & ? & 0 \end{pmatrix}$$

? = no additional base vector in E_0 $\Rightarrow$ A not diagonalizable.

3 Powers of matrices

$\underset{\substack{n\times n\\ \text{matrix}}}{A}$: Compute $A^N = \underbrace{A\cdot A\cdot A\cdot \ldots \cdot A}_{N \text{ times}}$ (N bis)

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $A^{37} = ?$

If A is diagonalizable:

P.1 $P^{-1}AP = D$

$AP = PD \quad | \cdot P^{-1}$

$A = PDP^{-1}$

$$A^N = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\ldots(\cancel{P}DP^{-1})}_{N \text{ times}}$$

$$A^N = PD^NP^{-1}$$

$$= P\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_n \end{pmatrix}^N P^{-1}$$

$$= P\begin{pmatrix} \lambda_1^N & & 0 \\ & \lambda_2^N & \\ 0 & \ddots & \lambda_n^N \end{pmatrix} P^{-1}$$

Ex: $A = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$

① Eigenvalues: $\begin{vmatrix} 0.7-\lambda & 0.2 \\ 0.3 & 0.8-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 1.5\lambda + 0.50 = 0$$

$$(\lambda - 1)(\lambda - 0.50) = 0$$

$$\underline{\lambda_1 = 1} \quad \underline{\lambda_2 = 0.50}$$ (A diagonalizable!)

② Eigenvectors:

$\lambda = 1$: $\begin{pmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{pmatrix} \to \begin{pmatrix} -0.3 & 0.2 \\ 0 & 0 \end{pmatrix}$ $\quad -0.3x + 0.2y = 0$, y free

$$x = \frac{0.2y}{0.3} = \frac{2y}{3}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y/3 \\ y \end{pmatrix} = \underline{\frac{y}{3}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

$$E_1 = \text{span}(\underline{v_1}), \quad \underline{v_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\lambda = 0.5$: $\begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{pmatrix} \to \begin{pmatrix} 0.2 & 0.2 \\ 0 & 0 \end{pmatrix}$ $\quad 0.2x + 0.2y = 0$, y free

$$x = \frac{-0.2y}{0.2} = -y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = \underline{y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$E_{0.5} = \text{span}(\underline{v_2}), \quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

③ Diagonalization:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \quad P = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \implies P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

$$A^N = PD^NP^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1^N & 0 \\ 0 & 0.5^N \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.5^N \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2 & -0.5^N \\ 3 & 0.5^N \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \underline{\frac{1}{5}\begin{pmatrix} 2+3\cdot 0.5^N & 2-2\cdot 0.5^N \\ 3-3\cdot 0.5^N & 3+2\cdot 0.5^N \end{pmatrix}}$$

$$\downarrow$$

as $N \to \infty$:

$$\frac{1}{5}\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{pmatrix}}}$$