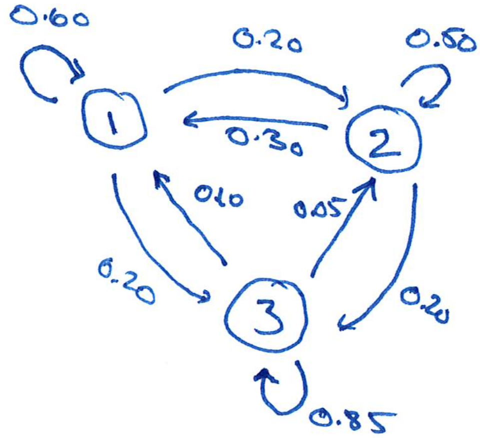
Plan

1. Application: Markov chains
2. Definiteness of quadratic forms
3. Orthogonal diagonalization

1 Application: Markov chains



Number of states: $n = 3$

State vector: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ $v_1, v_2, v_3 \geq 0$, $v_1 + v_2 + v_3 = 1$

v_i = share of population in state i

$\underline{v_t}$: state vector after t time periods

Ex: $\underline{v_0} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}$

Transition matrix: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $a_{ij} \geq 0$ for all i, j; the sum in each column is 1

a_{ij} = share of pop. in state j that transitions to state i in the next time period

$$\boxed{\underline{v_{t+1}} = A \cdot \underline{v_t}}$$

$$\Rightarrow \underline{v_T} = A^T \cdot \underline{v_0}$$

Ex:

$$A = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.6 & 0.3 & 0.10 \\ 0.2 & 0.5 & 0.05 \\ 0.2 & 0.2 & 0.85 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$\underline{v_0} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}$$

$$\underline{v_1} = \begin{pmatrix} 0.6 & 0.3 & 0.10 \\ 0.2 & 0.5 & 0.05 \\ 0.2 & 0.2 & 0.85 \end{pmatrix} \cdot \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3 + 0.15 + 0 \\ 0.1 + 0.25 + 0 \\ 0.1 + 0.1 + 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0.45 \\ 0.35 \\ 0.20 \end{pmatrix}}}$$

$$\underline{v_2} = A \cdot \underline{v_1} = A(A \cdot \underline{v_0}) = A^2 \cdot \underline{v_0}$$

Want to compute / find the marked vector (state vector) after T time periods when T is large / $T \to \infty$ = Long run

Method I: Compute A^T using eigenvalues / eigen vectors

$$\boxed{\lim_{T \to \infty} A^T \cdot \underline{v}_0}$$

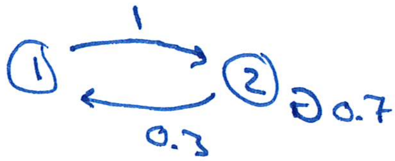
Method 2: Using Perron-Frobenius theory.

Defn: A Markov chain is called regular if $\underbrace{A^t > 0}_{\text{all entries of } A^t \text{ are } > 0}$ for some $t \geq 1$.

Ex: $A = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.05 \\ 0.2 & 0.2 & 0.85 \end{pmatrix}$ regular $(t=1)$

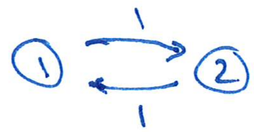
$$A = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{pmatrix} 0 & 0.3 \\ 1 & 0.7 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \quad A^2 = \begin{pmatrix} 0.3 & 0.21 \\ 0.7 & 0.79 \end{pmatrix}$$

regular $(t=2)$



A is regular $\Longleftrightarrow$ you can get from any node to any other node in exactly t steps.
$(A^t > 0)$

Ex: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ not regular



Thm (Perron-Frobenius)

If A is a regular Markov chain matrix, then

i) $\lambda = 1$ is an eigenvalue of mult $m=1$, and $|\lambda| < 1$ for all other eigenvalues

ii) there is a unique state vector $\underline{v}$ in E_1.

Moreover, $\lim_{t \to \infty} A^t \cdot \underline{v}_0 = \underline{v}$ for any initial state vector $\underline{v_0}$.

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Ex: $A = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.05 \\ 0.2 & 0.2 & 0.85 \end{pmatrix}$ regular Markov chain matrix

$\lambda = 1$: $\begin{pmatrix} -0.4 & 0.3 & 0.1 \\ 0.2 & -0.5 & 0.05 \\ 0.2 & 0.2 & -0.15 \end{pmatrix} \begin{matrix} \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \\ \end{matrix} \to \begin{pmatrix} -0.4 & 0.3 & 0.1 \\ 0 & -0.35 & 0.1 \\ \cancel{0} & \cancel{-0.35} & \cancel{0.1} \end{pmatrix}$

$$-0.4x + 0.3y + 0.1z = 0$$
$$-0.35y + 0.1z = 0$$

$\to y = \frac{0.1}{0.35} z = \frac{10}{35} z = \frac{2}{7} z$

$-0.4x = -0.3\left(\frac{2}{7}z\right) - 0.1z \quad | \cdot (-10)$

$4x = \frac{6}{7}z + z = \frac{13}{7} z$

$x = \frac{13}{28} z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13/28\, z \\ 2/7\, z \\ z \end{pmatrix} = \frac{z}{28} \cdot \begin{pmatrix} 13 \\ 8 \\ 28 \end{pmatrix}$$

$\underline{v}_1 = \begin{pmatrix} 13 \\ 8 \\ 28 \end{pmatrix}$ base of E_1

$13 + 8 + 28 = 49 \Rightarrow \underline{v} = \frac{1}{49}\begin{pmatrix} 13 \\ 8 \\ 28 \end{pmatrix} = \begin{pmatrix} 13/49 \\ 8/49 \\ 28/49 \end{pmatrix}$

Long run equilibrium market shares: $\begin{pmatrix} 13/49 \\ 8/49 \\ 28/49 \end{pmatrix} \simeq \begin{pmatrix} 0.265 \\ 0.163 \\ 0.571 \end{pmatrix}$

② Definiteness of quadratic forms

Defn: A polynomial function $q(x_1, x_2, \ldots, x_n)$ in n variables is a quadratic form if all terms have degree 2.

Ex: $q(x,y,z) = x^2 - 6xy + y^2 + 4yz + 3z^2$

Fact: $\boxed{q(\underline{x}) \text{ quadratic form in } n \text{ variables} \longleftrightarrow A \text{ symmetric } n\times n \text{ matrix}}$

$$q(\underline{x}) = \underline{x}^T A \underline{x}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Ex: $A = \begin{array}{cc} \begin{array}{cc} x & y \end{array} & \\ \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} & \begin{array}{c} x \\ y \end{array} \end{array}$: symm. $n\times n$ matrix ($n=2$)

$$q(\underline{x}) = \underline{x}^T A \underline{x} = (x \ y)\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= (x+2y \quad 2x+3y) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= (x+2y)\cdot x + (2x+3y)\cdot y$$
$$= x^2 + 2yx + 2xy + 3y^2$$
$$= \underline{x^2 + 4xy + 3y^2}$$

Ex: $q(x,y,z) = \underline{x^2} - 6xy + \underline{y^2} + 4yz + \underline{3z^2}$

$$A = \begin{array}{cc} \begin{pmatrix} 1 & -3 & 0 \\ -3 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \\ \begin{array}{ccc} x & y & z \end{array} & \end{array}$$

Fact: $q(\underline{x}) = q(x_1, x_2, \ldots, x_n)$ quadr. form in n var's.

i) $q(\underline{0}) = 0$

ii) $\underline{x} = \underline{0}$ is a stationary pt. of q. $(q'_{x_1} = q'_{x_2} = \cdots = q'_{x_n} = 0)$

Ex: $q(x,y) = x^2 + 4xy + 3y^2$ $\quad q'_x = 2x + 4y$, $\quad q'_y = 4x + 6y$

Defn. A quadr. form $q(\underline{x})$ is called

(1) positive semidefinite if $q(\underline{x}) \geq 0$ for all $\underline{x}$

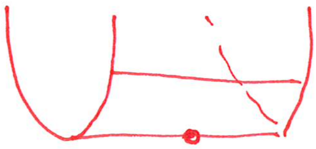
(2) negative —"— $\quad q(\underline{x}) \leq 0$ —"—

(3) ~~otherwise~~ indefinite otherwise (q has both pos. and neg. values)

(a) positive definite if $q(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$

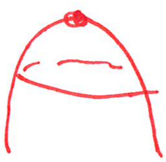
(b) negative —"— if $q(\underline{x}) < 0$ —"—

(a) and (1)

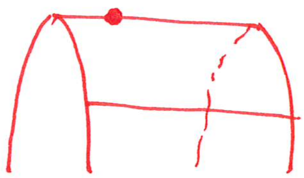


(1) not (a)

$\underline{x} = \underline{0}$ is global min for q

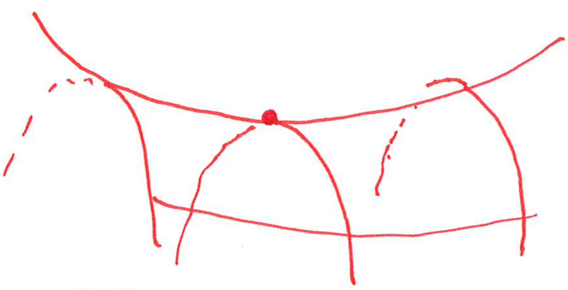


(b) and (2)



(2) not (b)

$\underline{x} = \underline{0}$ is global max



(3) $\quad \underline{x} = \underline{0}$ saddle pt.

Thm $q(\underline{x})$ quadr. fcn in n var's with symm. matrix A, and A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(1) A pos. semidefn. $\iff$ $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$
(2) A neg. — " — $\iff$ $\lambda_1, \lambda_2, \ldots, \lambda_n \leq 0$
(3) A indefinite $\iff$ there are both pos. and neg. eigenvalues

(a) A pos. defn. $\iff$ $\lambda_1, \ldots, \lambda_n > 0$
(b) A neg. " $\iff$ $\lambda_1, \ldots, \lambda_n < 0$

Alternative method: Using principal minors

Ex: $q = x^2 + 4xy + 3y^2$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

Leading principal minors

$D_1 = 1 \leftarrow M_{1,1}$

$D_2 = 1 \cdot 3 - 2 \cdot 2 = -1 \leftarrow M_{12,12}$

In general: A symm $n \times n$

$$\left.\begin{aligned} D_1 &= M_{1,1} \\ D_2 &= M_{12,12} \\ D_3 &= M_{123,123} \\ &\vdots \\ D_n &= M_{123\ldots n,123\ldots n} = |A| \end{aligned}\right\}$$

leading principal minors of A

$D_i =$ minor given by the first i rows and cols

Result: A symm $n \times n$ matrix

i) $D_1, D_2, \ldots, D_n > 0 \iff A$ positive defn.

ii) $(-1)^i D_i > 0 \iff$ $D_1 < 0, D_2 > 0, D_3 < 0, \ldots$ $\iff A$ negative defn.

Ex: $x^2 + y^2$ $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$D_1 = 1$, $D_2 = 1$

pos. defn.

Ex: $-2x^2 - 3y^2$ $A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$

$D_1 = -2$

$D_2 = (-2)(-3) = 6$

Summary:

A symm. $n \times n$ matrix
$$\begin{cases} D_i = M_{12\ldots i, 12\ldots i} \leftarrow \text{leading principal minor of order } i \\ \Delta_i = \text{any principal minor of order } i \end{cases}$$

Result:

(1) A pos. defn. $\Leftrightarrow$ $D_1, D_2, \ldots, D_n > 0$

(2) A neg. defn. $\Leftrightarrow$ $D_1 < 0, D_2 > 0, \ldots$

(3) A pos. semidefn. $\Leftrightarrow$ $\Delta_i \geq 0$ for all principal minors of order i

(4) A neg. semidefn. $\Leftrightarrow$ $\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots$ for all principal minors

(5) A indefinite $\Leftrightarrow$ otherwise

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $\quad D_1 = 1$, $D_2 = -1$ $\quad \Delta_1 = \underset{M_{1,1}}{1}, \underset{M_{2,2}}{3} \geq 0$, $\Delta_2 = -1$ $\quad$ indefinite

Ex: $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$ $\quad D_1 = 2$, $D_2 = 10$, $D_3 = 5 \cdot (4-9) = -25$ $\quad \Delta_1 = 2, 5, 2$, $\Delta_2 = 10, 10, -5$, $\Delta_3 = -25$

indefinite

Δ_i for i even is negative $\Rightarrow$ A indefinite

Δ_i for i odd with two different signs $\Rightarrow$ A indefinite

Reduced rank criterion (RRC)

A symm $n\times n$ - matrix

(1) $\text{rk}(A) = r < n$ and $D_1, D_2, \ldots, D_r > 0$
$\Rightarrow$ A positive semidefin.

(2) $\text{rk}(A) = r < n$ and $(-1)^i D_i > 0$ for $i = 1, \ldots, r$
$\Rightarrow$ A negative semidefin.

Ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 8 \end{pmatrix}$ $D_1 = 2$, $D_2 = 7$, $D_3 = |A| = 0$

$\boxed{\begin{array}{l} \text{rk}(A) = 2 \\ D_1 = 2,\ D_2 = 7 > 0 \end{array}}$

$\Rightarrow$ A pos. semidefin. by the reduced rank crit.

③ Orthogonal diagonalization

Defn. P is called orthogonal if $P^{-1} = P^T$

Defn. An orthogonal diagonalization of A is $P^{-1}AP = D$

A $n\times n$ matrix

where D is diagonal and P is orthogonal, i.e

$$\boxed{P^T A P = D}$$

Note: Special case $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$, $P = \begin{pmatrix} \underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n \end{pmatrix}$

but must choose P such that $P^{-1} = P^T$.

Thm: There is an orthogonal diagonalization of A if and only if A is symmetric.

How do we find an orthogonal diag. of A when A is symmetric?

$P^{-1} = P^T \Leftrightarrow P^T \cdot P = I$

$P = (\underline{v_1} | \underline{v_2} | \cdots | \underline{v_n})$:

$$\begin{pmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \end{pmatrix} \cdot \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right) = I$$

P orthogonal

$\Updownarrow$

i) $\underline{v_i} \cdot \underline{v_i} = 1$ for $i = 1, 2, \dots n$.

($\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ have length 1)

ii) $\underline{v_i} \cdot \underline{v_j} = 0$ for $i \neq j$

(orthogonal vectors)

$$\begin{pmatrix} \underline{v_1} \cdot \underline{v_1} & \underline{v_1} \cdot \underline{v_2} & \cdots \\ \underline{v_2} \cdot \underline{v_1} & \underline{v_2} \cdot \underline{v_2} & \cdots \\ \vdots & \vdots & \end{pmatrix} = \underline{I}$$

Fact: * If $\underline{v}$ is an eigenvector in E_λ, then

$\frac{1}{\|\underline{v}\|} \cdot \underline{v}$ is an eigenvector in E_λ of length 1.

* If $\underline{v}, \underline{w}$ are eigenvectors with different eigenvalues then $\underline{v} \cdot \underline{w} = 0$

Ex. 1 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ symm.

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1, \lambda = 3$

$\lambda = 1$: $\begin{pmatrix} 1 & 1 \\ \cancel{1} & \cancel{1} \end{pmatrix} \to v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} -1 & 1 \\ \cancel{1} & \cancel{-1} \end{pmatrix} \to v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ordinary diag: $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

Note: $v_1 \cdot v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1 \cdot 1 + 1 \cdot 1 = 0$ (since $\lambda_1 = 1 \neq \lambda_2 = 3$)

$\|v_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \to v_1' = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\|v_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\quad v_2' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Orthogonal diag.: $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \frac{1}{-1} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = P^T$

Ex. 2 $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$\lambda = -1$ eigenvalue of mult. $m \geq 2$ since $A - \lambda I = A + I = \begin{pmatrix} 1 & 1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{1} \end{pmatrix}$ has rk = 2 $\Rightarrow \dim E_{-1} = 2 \leq m$

$\lambda_1 = \lambda_2 = -1$

$(-1) + (-1) + \lambda_3 = tr(A) = 0 \Rightarrow \lambda_3 = 2$

$\lambda = -1$: $\begin{pmatrix} 1 & 1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{1} \end{pmatrix}$ $x + y + z = 0$, y, z free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 2 \\ 0 & -3 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$ $\quad -x - y + 2z = 0 \quad x = z$; $\quad -3y + 3z = 0 \quad y = z$; $\quad z$ free

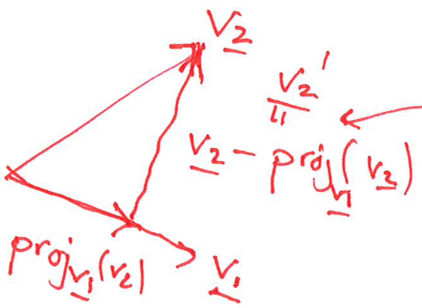
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Diag:
$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\underline{v_1} \cdot \underline{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$$

$$\underline{v_1} \cdot \underline{v_3} = \underline{v_2} \cdot \underline{v_3} = 0 \quad \text{since } \underset{\overset{\shortparallel}{\lambda_2}}{\lambda_1} = -1 \neq \lambda_3 = 2$$



$$\underline{v_2}' = \underline{v_2} - \text{proj}_{\underline{v_1}}(\underline{v_2})$$
$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{\underline{v_1} \cdot \underline{v_2}}{\underline{v_1} \cdot \underline{v_1}} \cdot \underline{v_1}$$
$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\left.\begin{aligned} \underline{v_1} &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \underline{v_2}' &= \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \to \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \\ \underline{v_3} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \to \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}\right\} \quad \underline{P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}} \quad \underline{D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$