

Plan

- 1 Constrained optimization
- 2 Lagrange problems
- 3 Second order conditions

- plenary session 2 : Mon

- midterm exam (week)

Fri Oct 06 - Thu Oct 12

- midterm evaluation

Review:

Unconstrained optimization: $\max/\min f(x_1, x_2, \dots, x_n) = f(\underline{x})$

① Stationary pts: $f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0$ ← Candidates for max/min

② Second derivative test: $\underline{x}^*$ stationary pt. local data

$H(f)(\underline{x}^*)$ pos. defn $\Rightarrow \underline{x}^*$ local min
 neg. defn. $\Rightarrow \underline{x}^*$ local max
 indefinite $\Rightarrow \underline{x}^*$ saddle pt

if pos. semidefn. but not pos. defn $\Rightarrow$ test inconclusive
 (neg. —) neg || — || —

③ Convex/concave:

f convex $\Leftrightarrow H(f)(\underline{x})$ pos. semidefn for all $\underline{x}$ global data $\Rightarrow$ all stat. pts are global min.

f concave $\Leftrightarrow H(f)(\underline{x})$ neg. " $\Rightarrow$ all stat. pts. are global max

④ If $f(\underline{x}) = \underline{x}^T A \underline{x} + B \underline{x} + C$:

(a) $f'(\underline{x}) = \begin{pmatrix} f'_{x_1} \\ \vdots \\ f'_{x_n} \end{pmatrix} = 2A \underline{x} + B^T$ (b) $H(f) = 2A$

① Constrained optimization

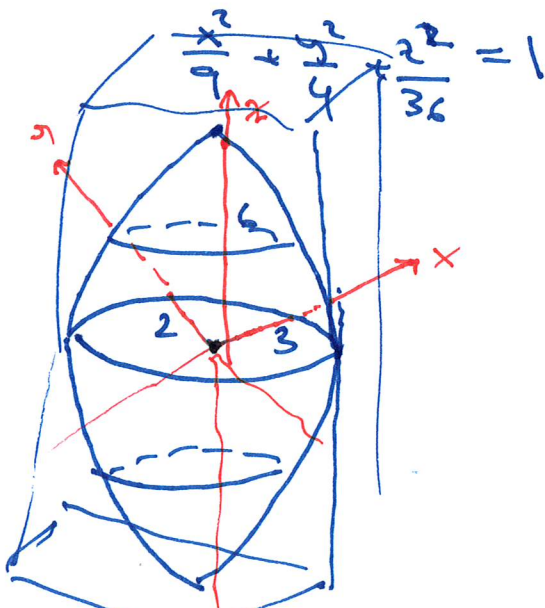
Typical example: $\max/\min \underbrace{f(x_1, \dots, x_n)}_{\text{obj. function}}$ when $\dots$ $\underbrace{\text{Constraints} = \text{one or more equations or in equalities}}$

Ex: $\max/\min \underbrace{f(x, y, z) = x + y + z}_{\text{obj. fn.}}$ when $\underbrace{4x^2 + 9y^2 + z^2 = 36}_{\text{constraint}}$

Notation:

The set D of admissible points = all points that satisfy all constraints

$$D: 4x^2 + 9y^2 + z^2 = 36 \quad 1: 36$$

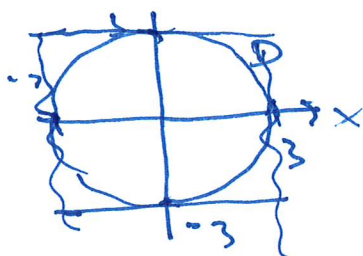


compact $\left\{ \begin{array}{l} \text{ellipsoid} \\ \text{closed} \\ \text{bounded} \end{array} \right. \left\{ \begin{array}{l} -3 \leq x \leq 3 \\ -2 \leq y \leq 2 \\ -6 \leq z \leq 6 \end{array} \right.$

Defn: The set D is closed if all boundary pts of D are included in D . Closed: $\leq \geq =$

Defn: The set D is bounded if there are constants $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that $a_1 \leq x_1 \leq b_1$
 $a_2 \leq x_2 \leq b_2$
 $\vdots$
 $a_n \leq x_n \leq b_n$
 for all points $(x_1, \dots, x_n)$ in D .

Ex: $x^2 + y^2 = 9$

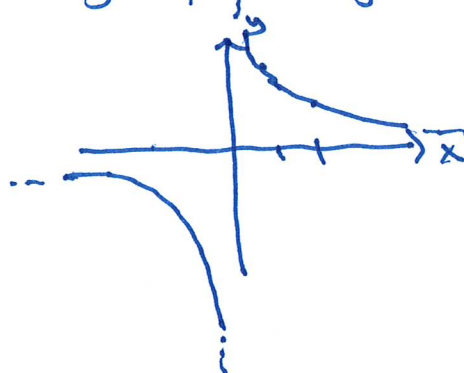


$$-3 \leq x \leq 3$$

$$-3 \leq y \leq 3$$

bounded

$xy = 4 \Leftrightarrow y = 4/x$



not bounded

Defn: D is called compact if it is closed and bounded

Thm (Extreme Value Thm = EVT)

If f is a cont. function on a compact set D , then f has a max and a min on D .

② Lagrange problems: Constrained opt. problem with equality constraints.

General form:

$$\begin{pmatrix} n = \# \text{var's} \\ m = \# \text{constraints} \end{pmatrix} \quad \begin{matrix} \max/\min f(x_1, x_2, \dots, x_n) \\ \text{where } f(x) \end{matrix} \quad \text{when} \quad \begin{cases} g_1(x) = a_1 \\ g_2(x) = a_2 \\ \vdots \\ g_m(x) = a_m \end{cases}$$

$\lambda = \text{lambda} = \text{greek } \lambda$ Method of Lagrange multipliers:

$$L = L(x_1, x_2, \dots, x_n; \lambda_1, \lambda_2, \dots, \lambda_m) = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \lambda_2 \cdot (g_2(\underline{x}) - a_2) - \dots - \lambda_m \cdot (g_m(\underline{x}) - a_m)$$

Variables
Lagrange multipliers

Lagrangian function

$L'_{x_1} = 0$ $L'_{x_2} = 0$ $\vdots$ $L'_{x_n} = 0$	$g_1(\underline{x}) = a_1$ $g_2(\underline{x}) = a_2$ $\vdots$ $g_m(\underline{x}) = a_m$
<u>FOC</u>	<u>C</u>

Lagrange conditions:

FOC + C m+n equ's in
m+n var's

Candidates for max/min

Note: $L'_{x_i} = 0 - 1 \cdot (g_i(\underline{x}) - a_i) = 0$
 $\Leftrightarrow g_i(\underline{x}) = a_i$ constraint #i

Ex: max/min $f = x + y + z$ when $4x^2 + 9y^2 + z^2 = 36$

$$h = x + y + z - \lambda (4x^2 + 9y^2 + z^2 - 36)$$

$n=3$
 $m=1$ $m+n=4$

FOC $\begin{cases} L'_x = 1 - \lambda \cdot 8x = 0 \\ L'_y = 1 - \lambda \cdot 18y = 0 \\ L'_z = 1 - \lambda \cdot 2z = 0 \end{cases}$ C $4x^2 + 9y^2 + z^2 = 36$	$x = \frac{1}{8\lambda}$ $y = \frac{1}{18\lambda}$ $z = \frac{1}{2\lambda}$	$4\left(\frac{1}{8\lambda}\right)^2 + 9\left(\frac{1}{18\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 36$ $\left(\frac{1}{2\lambda}\right)^2 \cdot \left[4 \cdot \left(\frac{1}{4}\right)^2 + 9 \cdot \left(\frac{1}{9}\right)^2 + 1\right] = 36$ $\left(\frac{1}{2\lambda}\right)^2 \cdot \left(\frac{1}{4} + \frac{1}{9} + 1\right) = 36$
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$$\frac{1}{2\lambda} = \pm \frac{36}{\frac{1}{4} + \frac{1}{9} + 1} \quad 1 \cdot 2 \quad \frac{1}{\lambda} = \pm \frac{72}{7}$$

$$x = \frac{1}{8} \left(\pm \frac{72}{7} \right) = \pm \frac{9}{7} \quad y = \frac{1}{18} \left(\pm \frac{72}{7} \right) = \pm \frac{4}{7}$$

$$\left(\frac{1}{2\lambda}\right)^2 \cdot \frac{9+4+36}{36} = 36$$

$$\left(\frac{1}{2\lambda}\right)^2 = \frac{36 \cdot 36}{49} = \frac{36^2}{7^2}$$

$$z = \frac{1}{2} \left(\pm \frac{72}{7} \right) = \pm \frac{36}{7}$$

$$\lambda = \frac{7}{72} : x = 9/7, y = 4/7, z = 36/7 \rightarrow \left(\frac{9}{7}, \frac{4}{7}, \frac{36}{7}; \frac{7}{72} \right)$$

$$\lambda = -\frac{7}{72} : \rightarrow \left(-\frac{9}{7}, -\frac{4}{7}, -\frac{36}{7}; -\frac{7}{72} \right)$$

Conclusion: Two cand. pts
(so far) that satisfy FOCs: $\left(\frac{9}{7}, \frac{4}{7}, \frac{36}{7}; \frac{7}{72} \right) \quad f = 7$
 $\left(-\frac{9}{7}, -\frac{4}{7}, -\frac{36}{7}; -\frac{7}{72} \right) \quad f = -7$

EVT: $f = x + y + z$ cont.
 D closed and bounded } there is a max and a min of f on D

$$\Rightarrow \begin{aligned} f_{\max} &= \underline{\underline{7}} \quad \text{at } (x, y, z) = \left(\frac{9}{7}, \frac{4}{7}, \frac{36}{7} \right) \text{ with } \lambda = \frac{7}{72} \\ f_{\min} &= \underline{\underline{-7}} \quad \text{---} \quad \left(-\frac{9}{7}, -\frac{4}{7}, -\frac{36}{7} \right) \text{ with } \lambda = -\frac{7}{72} \end{aligned}$$

Technical condition: NDCQ = non-degenerate constraint qualification

$$\begin{aligned} C: \quad g_1(x) &= a_1 \\ g_2(x) &= a_2 \\ &\vdots \\ g_m(x) &= a_m \end{aligned}$$

Defn: NDCQ: $\boxed{rk J = m}$

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \dots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} \quad \left. \vphantom{\begin{pmatrix} \frac{\partial g_1}{\partial x_1} \\ \vdots \\ \frac{\partial g_m}{\partial x_1} \end{pmatrix}} \right\} m$$

Ex:

$$\begin{aligned} 4x^2 + 9y^2 + z^2 &= 36 : \\ 0 &\neq 36 \end{aligned}$$

NDCQ:

$$rk \begin{pmatrix} 8x & 18y & 2z \end{pmatrix} = 1$$

NDCQ fails:

$$\begin{aligned} 8x &= 0 \quad 18y = 0 \quad 2z = 0 \\ x &= 0, y = 0, z = 0 \end{aligned}$$

Exceptional candidate pts : adm. pts where NDCQ fails

$$\boxed{C + \nabla h(z) < 0}$$

Ex:

$$4x^2 + 9y^2 + z^2 = 36$$

$$\nabla h(8x, 18y, 2z) = 0$$

$$0 \neq 36$$

$$8x = 18y = 2z = 0$$

$$(x, y, z) = (0, 0, 0)$$

No adm. pts where NDCQ fails

= All adm. pts satisfy NDCQ

Method:

① Find all cand. pts

= candidate for max/min

- ordinary cand pts : $\boxed{FOC+C}$

- exceptional cand pts:

$$\boxed{C + \nabla h(z) < 0}$$

② Determine if the best cand pts are max/min.

- Use EVT if D is bounded.

- Use SOC = second order condition

Thm (Necessary cond. for max/min)

If x^* is a max/min in a Lagrange problem, and x^* satisfy NDCQ, then there are Lagrange multipliers λ^* such that (x^*, λ^*) satisfy FOC+C.

③ Second order condition (SOC)

Thm. (SOC)

If $(\underline{x}^*, \underline{\lambda}^*)$ satisfies FOC + C, then we consider $h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$:

h convex $\Rightarrow \underline{x}^*$ is a minimum point

h concave $\Rightarrow \underline{x}^*$ is a maximum point

Ex: max $f = x + y + z$ when $4x^2 + 9y^2 + z^2 = 36$

$$h = x + y + z - \lambda(4x^2 + 9y^2 + z^2 - 36)$$

$$\begin{aligned} h'_x &= 1 - \lambda \cdot 8x = 0 \\ h'_y &= 1 - \lambda \cdot 18y = 0 \\ h'_z &= 1 - \lambda \cdot 2z = 0 \\ 4x^2 + 9y^2 + z^2 &= 36 \end{aligned}$$

FOC + C

$$(\underline{x}, \underline{y}, \underline{z}; \underline{\lambda}) = (1/7, 4/7, 36/7; 7/72)$$

$$h = \underbrace{x + y + z} - \frac{7}{72} \underbrace{(4x^2 + 9y^2 + z^2 - 36)}$$

$$H(h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{7}{72} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

neg. defn. at all x, y, z

pos. defn.

$\lambda_1 = 8 \quad \lambda_2 = 18 \quad \lambda_3 = 2 > 0$

h concave $\cap$

$$f_{\max} = f(1/7, 4/7, 36/7) = \underline{\underline{7}}$$

Ex: $\max/\min f = xw - yz$ when $\begin{cases} x^2 + 4y^2 = 4 \\ 4z^2 + 9w^2 = 36 \end{cases}$

$$L = xw - yz - \lambda_1(x^2 + 4y^2 - 4) - \lambda_2(4z^2 + 9w^2 - 36)$$

$$(1) L'_x = w - \lambda_1 \cdot 2x = 0$$

$$(2) L'_y = -z - \lambda_1 \cdot 8y = 0$$

$$(3) L'_z = -y - \lambda_2 \cdot 8z = 0$$

$$(4) L'_w = x - \lambda_2 \cdot 18w = 0$$

$$(5) x^2 + 4y^2 = 4$$

$$(6) 4z^2 + 9w^2 = 36$$

$$w = 2\lambda_1 x \quad (1)$$

$$z = -8\lambda_1 y \quad (2)$$

$$-y - 8\lambda_2(-8\lambda_1 y) = 0 \quad (3)$$

$$x - 18\lambda_2(2\lambda_1 x) = 0$$

$$x(1 - 36\lambda_1\lambda_2) = 0$$

$$(1)+(4): \boxed{x=0 \text{ or } \lambda_1\lambda_2 = 1/36}$$

$$-y(1 - 64\lambda_1\lambda_2) = 0$$

Four cases (a) - (d) based on $\begin{matrix} (1)+(4): \\ (2)+(3): \end{matrix} \rightarrow \boxed{y=0 \text{ or } \lambda_1\lambda_2 = 1/64}$

(a) $x=0, y=0$: $w=0$ from (1)
 $z=0$ from (2) $\left. \vphantom{\begin{matrix} w=0 \\ z=0 \end{matrix}} \right\} (x,y,z,w) = (0,0,0,0)$
 does not satisfy C (5) and (6)

(b) $x=0, \lambda_1\lambda_2 = 1/64$: $w=0$ from (1)

$$(5) x^2 + 4y^2 = 4y^2 = 4 \Rightarrow y^2 = 1 \quad y = \pm 1$$

$$(6) 4z^2 + 9w^2 = 4z^2 = 36 \quad z^2 = 9 \quad z = \pm 3$$

$$(2) \lambda_1 = -\frac{z}{8y} = \pm \frac{3}{8}$$

$$\lambda_1\lambda_2 = \frac{1}{64}: \lambda_2 = \pm \frac{1}{24}$$

$$\frac{1}{64}: \frac{3}{8} = \frac{1}{64} \cdot \frac{8}{3} = \frac{1}{8 \cdot 3} = \frac{1}{24}$$

Candidate pts:

$$(0, 1, 3, 0; -3/8, -1/24) \quad f = -3$$

$$(0, -1, -3, 0; -3/8, -1/24) \quad f = -3$$

$$(0, 1, -3, 0; 3/8, 1/24) \quad f = 3$$

$$(0, -1, 3, 0; 3/8, 1/24) \quad f = 3$$

(c) $y=0, \lambda_1 \lambda_2 = \frac{1}{36}$: $z=0$ from (2)

(5) $x^2=4 \quad x=\pm 2$

(6) $9w^2=36 \quad w^2=4 \quad w=\pm 2$

(11) $\lambda_1 = \frac{1}{2x} = \pm \frac{1}{2}$

$\lambda_1 \lambda_2 = \frac{1}{36}$: $\lambda_2 = \pm \frac{1}{18}$

$\frac{1}{36} : \frac{1}{2} = \frac{1}{36} \cdot \frac{2}{1} = \frac{1}{18}$

Cand. pts:

$(2, 0, 0, 2; \frac{1}{2}, \frac{1}{18}) \quad f=4$

$(2, 0, 0, -2; -\frac{1}{2}, -\frac{1}{18}) \quad f=-4$

$(-2, 0, 0, 2; \frac{1}{2}, \frac{1}{18}) \quad f=4$

$(-2, 0, 0, -2; -\frac{1}{2}, -\frac{1}{18}) \quad f=-4$

(d) $\lambda_1 \lambda_2 = \frac{1}{36}, \lambda_1 \lambda_2 = \frac{1}{64}$: impossible

SOC on $(2, 0, 0, 2)$ $\lambda_1 = \frac{1}{2} \quad \lambda_2 = \frac{1}{18} \quad h = xw - yz - \frac{1}{2}(x^2 + 4y^2 - 4) - \frac{1}{18}(4z^2 + 2w^2 - 36)$

$f_{\max} = f(2, 0, 0, 2) = 4$
 $= f(-2, 0, 0, -2) = 4$

$H(h) = 2 \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -\frac{2}{9} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}$

Similar computation
using SOC:

$f_{\min} = f(-2, 0, 0, 2) = -4$
 $= f(2, 0, 0, -2) = -4$

RRC:

$\text{rk } A = 3$

A is
neg. semidefn.

$\Downarrow$

$H(h) = 2A$ neg. semidefn.

$\Downarrow$

h concave

$\Downarrow$

$(2, 0, 0, 2)$ is max.

A:

$D_1 = -42 < 0$

$D_2 = 1 > 0$

$D_3 = -\frac{1}{2}(4/9 - 1/4)$

$= -\frac{1}{2} \left(\frac{16-9}{36} \right) < 0$

$b_4 = 0$