Plan

1 Kuhn-Tucker problems
2 Envelope theorems

(1) Kuhn-Tucker problems = optimization problems with closed inequality constraints ($\leq$ or $\geq$)

Ex: max $f(x,y,z) = x+y+z$ when $4x^2+9y^2+z^2 \leq 36$ (1)

(a) Extreme value theorem

$D = \{(x,y,z) : 4x^2+9y^2+z^2 \leq 36\}$
set of adm pts

$-3 \leq x \leq 3$
$-2 \leq y \leq 2$
$-6 \leq z \leq 6$

closed (V)
bounded (V)
$\Downarrow$ EVT
there is a max and a min in (1)

(b) Candidate points

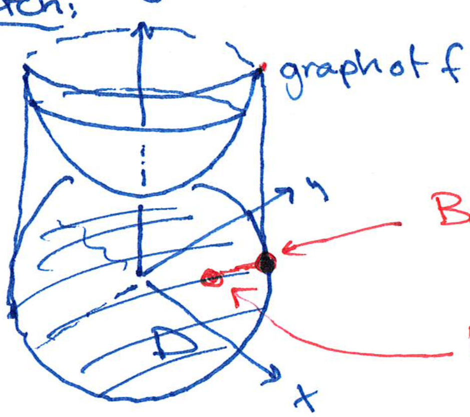
$\mathcal{L} = x+y+z - \lambda(4x^2+9y^2+z^2-36)$

$$\underbrace{\begin{array}{l} \mathcal{L}'_x = 1-\lambda\cdot 8x = 0 \\ \mathcal{L}'_y = 1-\lambda\cdot 18y = 0 \\ \mathcal{L}'_z = 1-\lambda\cdot 2z = 0 \end{array}}_{\text{FOC}} \qquad \underbrace{4x^2+9y^2+z^2 \leq 36}_{\text{C}} \qquad \underbrace{\begin{array}{l} \lambda \geq 0 \\ \text{and} \\ \lambda(4x^2+9y^2+z^2-36) = 0 \end{array}}_{\text{CSC}}$$

CSC = complementary slackness cond.

		CSC
Binding:	$4x^2+9y^2+z^2=36$	$\lambda \geq 0$
	or	
Non-binding	$4x^2+9y^2+z^2<36$	$\lambda = 0$

Sketch:



Boundary pts: $x^2+y^2=16$ Binding constraint

Interior pts: $x^2+y^2<16$ Non-binding constraint

max/min
$\Downarrow$
Stationary pt for f
$\Downarrow$
$\lambda = 0$

$C: \; x^2+y^2 \leq 16$

Candidate pts in (1)

Binding case	Non-binding case
C: $4x^2+9y^2+z^2=36$	C: $4x^2+9y^2+z^2<36$
FOC: $1-\lambda\cdot 8x=0$, $1-\lambda\cdot 18y=0$, $1-\lambda\cdot 2z=0$	FOC: $1-\lambda\cdot 8x=0$, $1-\lambda\cdot 18y=0$, $1-\lambda\cdot 2z=0$
CSC: $\lambda\geq 0$	CSC: $\underline{\lambda=0}$
See last lecture: $(x,y,z;\lambda)=\underline{(9/7,4/7,36/7;7/72)}$ f=7	no solutions
$\cancel{(-9/7,-4/7,-36/7;-7/72)}$	(no cand. pts in this case)

(C) Second order condition

If $(\underline{x}^*;\lambda^*)$ is an ordinary candidate pt (that satisfies FOC+C+CSC), then we look at $h(\underline{x})=L(\underline{x};\lambda^*)$:

h concave $\Rightarrow \underline{x}^*$ is max

(h convex $\Rightarrow \underline{x}^*$ is min)

Ex: $h=L(x,y,z;7/72)=x+y+z-\frac{7}{72}(\underline{4x^2+9y^2+z^2}-36)$

$$H(h)=\underbrace{-\frac{7}{72}\begin{pmatrix}8&0&0\\0&18&0\\0&0&2\end{pmatrix}}_{\text{negative def.}}\Rightarrow h \text{ concave} \overset{\text{SOC}}{\Rightarrow} (x,y,z)=(9/7,4/7,36/7)$$

is max. pt, and

$f_{max}=f(9/7,4/7,36/7)=\underline{\underline{7}}$

Defn A general Kuhn-Tucker problem in n variables with m constraints is in <u>standard form</u> if it can be written

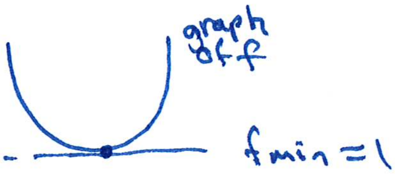
$$\max f(x_1,\dots,x_n) \quad \text{when} \begin{cases} g_1(x_1,\dots,x_n) \le a_1 \\ g_2(x_1,\dots,x_n) \le a_2 \\ \vdots \\ g_m(x_1,\dots,x_n) \le a_m \end{cases}$$

max, not min

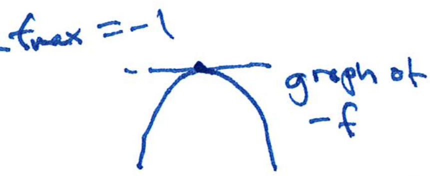
$\le$ and not $\ge$

Note: – Ex (1) was in std form
– Any problem (KT-problem) can be rewritten in std form

Ex: $\min f = 2x^2+y^2+3z^2$ when $\begin{cases} x-y+2z \ge 3 \\ x+y \ge 3 \end{cases}$



$$\begin{aligned} x-y+2z &\ge 3 \quad |\cdot(-1) \\ x+y &\ge 3 \quad |\cdot(-1) \end{aligned} \qquad \boxed{\begin{aligned} -x+y-2z &\le -3 \\ -x-y &\le -3 \end{aligned}}$$



$$\max -f = -2x^2-y^2-3z^2 \quad \text{when} \begin{cases} -x+y-2z \le -3 \\ -x-y \le -3 \end{cases}$$

std. form

minimum pt for f
= maximum pt for $-f$

$$\max f(\underline{x}) \quad \text{when} \quad \begin{cases} g_1(x) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$$

KT problem in std form

Method for finding candidate pts:

$$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \dots - \lambda_m (g_m(\underline{x}) - a_m)$$

FOC: $\mathcal{L}'_{x_1} = 0$, $\vdots$, $\mathcal{L}'_{x_n} = 0$

C: $g_1(\underline{x}) \le a_1$, $\vdots$, $g_m(\underline{x}) \le a_m$

CSC: $\lambda_1 \ge 0 \quad \lambda_1(g_1(\underline{x}) - a_1) = 0$, $\vdots$, $\lambda_m \ge 0 \quad \lambda_m(g_m(\underline{x}) - a_m) = 0$

Kuhn-Tucker conditions: FOC + C + CSC

Solutions FOC+C+CSC $\rightsquigarrow$ ordinary candidate pts

Ex: $\max f = -2x^2 - y^2 - 3z^2$ when $\begin{cases} -x+y-2z \le -3 \\ -x-y \le -3 \end{cases}$ (2)

std. form

$$\mathcal{L} = -2x^2 - y^2 - 3z^2 - \lambda_1(-x+y-2z+3) - \lambda_2(-x-y+3)$$
$$= -2x^2 - y^2 - 3z^2 + \lambda_1(x-y+2z-3) + \lambda_2(x+y-3)$$

FOC:
$$\mathcal{L}'_x = -4x + \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_y = -2y - \lambda_1 + \lambda_2 = 0$$
$$\mathcal{L}'_z = -6z + 2\lambda_1 = 0$$

C: $x - y + 2z \ge 3$, $x + y \ge 3$

CSC: $\lambda_1 \ge 0 \quad \lambda_1(x-y+2z-3) = 0$, $\lambda_2 \ge 0 \quad \lambda_2(x+y-3) = 0$

Case A:

$$\left.\begin{aligned} x-y+2z&=3 \quad B\\ x+y\quad&=3 \quad B\end{aligned}\right\} C$$

$$\left.\begin{aligned} -4x+\lambda_1+\lambda_2&=0\\ -2y-\lambda_1+\lambda_2&=0\\ -6z+2\lambda_1\quad&=0\end{aligned}\right\} \text{FOC}$$

$$\left.\begin{aligned} \lambda_1&\geq 0\\ \lambda_2&\geq 0\end{aligned}\right\} \text{CSC}$$

$$\underline{x=\frac{\lambda_1+\lambda_2}{4}} \quad \underline{y=\frac{-\lambda_1+\lambda_2}{2}} \quad z=\frac{2\lambda_1}{6}=\underline{\frac{\lambda_1}{3}} \qquad \underline{x=2} \quad \underline{y=1} \quad \underline{z=1}$$

(C1) $\dfrac{\lambda_1+\lambda_2}{4}-\dfrac{-\lambda_1+\lambda_2}{2}+2\dfrac{\lambda_1}{3}=3 \quad |\cdot 12$

$$3(\lambda_1+\lambda_2)-6(-\lambda_1+\lambda_2)+8\lambda_1=36$$

$$\boxed{17\lambda_1-3\lambda_2=36}$$

(C2) $\dfrac{\lambda_1+\lambda_2}{4}+\dfrac{-\lambda_1+\lambda_2}{2}=3 \quad |\cdot 4$

$$\lambda_1+\lambda_2+2(-\lambda_1+\lambda_2)=12$$

$$\boxed{-\lambda_1+3\lambda_2=12}$$

C(1)+(2): $\dfrac{16\lambda_1}{16}=\dfrac{48}{16}$ $\quad \underline{\lambda_1=3}$

$$-3+3\lambda_2=12 \quad \frac{3\lambda_2}{3}=\frac{15}{3} \quad \underline{\lambda_2=5}$$

Cond: $(x,y,z;\lambda_1,\lambda_2) = (2,1,1;3,5)$

cond. pts in case A. $-f = -12$

$f = \underline{x}^T A \underline{x} + Bx + C$

$f' = 2Ax + B^T$

$H(f) = 2A$

Try:

SOC on $(2,1,1;3,5)$: $h = L(x,y,z;3,5)$

$$= -2x^2 - y^2 - 3z^2 + 3(x-y+2z-3) + 5(x+y-3)$$

$-f_{max} = -12 \overset{\text{SOC}}{\Leftarrow} h$ concave $\Leftarrow H(h) = 2\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

A

neg. defn.

$-f(2,1,1)$

at $(x,y,z) = (2,1,1)$

with $(\lambda_1 = 3, \lambda_2 = 5)$

$\Downarrow$

$f_{min} = \underline{\underline{12}}$ at $(x,y,z) = \underline{\underline{(2,1,1)}}$ with $\lambda_1 = 3, \lambda_2 = 5$

d NDCQ for Kuhn-Tucker problems

C:

$$\boxed{\begin{matrix} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{matrix}}$$

$$J = \begin{pmatrix} \partial g_1/\partial x_1 & \cdots & \partial g_1/\partial x_n \\ \vdots & & \vdots \\ \partial g_m/\partial x_1 & \cdots & \partial g_m/\partial x_n \end{pmatrix}$$

$\underline{x}^*$ adm. pt

$J(\underline{x}^*)$ = all the rows in J where the constraints are binding at $\underline{x}^*$.

NDCQ: rk $J(\underline{x}^*)$ is maximal

Ex 2: $x-y+2z \geq 3$ $\quad J = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

$x+y \geq 3$

Case A: B-B (= =) NDCQ: $\text{rk}\left(\boxed{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}} \begin{matrix} 2 \\ 0 \end{matrix}\right) = 2$ ok

Case B: NB-B (> =) $\text{rk}(1 \; 1 \; 0) = 1$ ok

Case C: B-NB (= >) $\text{rk}(1 \; -1 \; 2) = 1$ ok

Case D: NB-NB (> >) no condition ok

2 Envelope theorems

- unconstrained
- Lagrange case
- Kuhn-Tucker case

Ex:

$$\max f = xw - yz \text{ when } \begin{cases} x^2+4y^2 = 4 \\ 4z^2+9w^2 = 36 \end{cases}$$

Solution: $f_{max} = \underline{\underline{4}}$ at $(2,0,0,2)$, $(-2,0,0,-2)$ with $\lambda_1 = 1/2$, $\lambda_2 = 1/18$

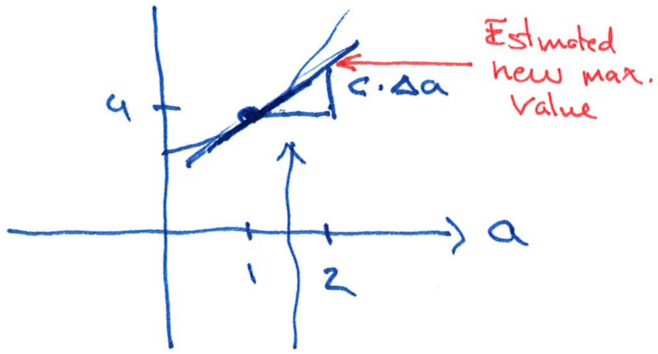
What about: $\max \; \underline{\underline{2}}xw - yz$ when $\begin{cases} x^2+4y^2=4 \\ 4z^2+9w^2=36 \end{cases}$?

Approach: $\max \; axw - yz$ when $\begin{cases} x^2+4y^2=4 \\ 4z^2+9w^2=36 \end{cases}$ Lagrange problem with parameter a

Notation: $f^*(a)$ = max value when the parameter is a $\quad f^*(1) = 4$, $f^*(2) = ?$

f^* = optimal value function

$$L = axw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$$



$\Delta a = 2 - 1 = 1$

If I know $f^*(a)$ at $a=1$, the tangent of $f^*(a)$ with slope $\frac{df^*(a)}{da}(a=1) = c$, then:

$$f^*(2) \simeq \underbrace{f^*(1)}_{=4} + \underbrace{c}_{\frac{df^*(a)}{da}(a=1)} \cdot \underbrace{(2-1)}_{\Delta a}$$

$$= 4 + 4 \cdot 1 = \underline{\underline{8}}$$

$$\boxed{\begin{array}{l} \underline{\text{Envelope thm}} \quad \text{(Lagrange case)} \\ \dfrac{df^*(a)}{da} = \dfrac{\partial L}{\partial a}\left(\underline{x}^*(a); \lambda^*(a)\right) \end{array}}$$

$\underline{a=1}$:

$x^*(1) =$	2	or -2	$\lambda_1^*(1) = 1/2$
$y^*(1) =$	0	0	$\lambda_2^*(1) = 1/18$
$z^*(1) =$	0	0	
$w^*(1) =$	2	-2	

$$\frac{\partial L}{\partial a} = xw \qquad \underbrace{\frac{\partial L}{\partial a}(\underline{x}^*(1); \lambda^*(1))}_{= x^*(1) \cdot w^*(1)} = 2 \cdot 2 = \underline{\underline{4}}$$

or $(-2) \cdot (-2) = 4$