Plan

1 Envelope Theorems
2 Quadratic functions and constrained optimization
2 Minimum variance portfolio problems

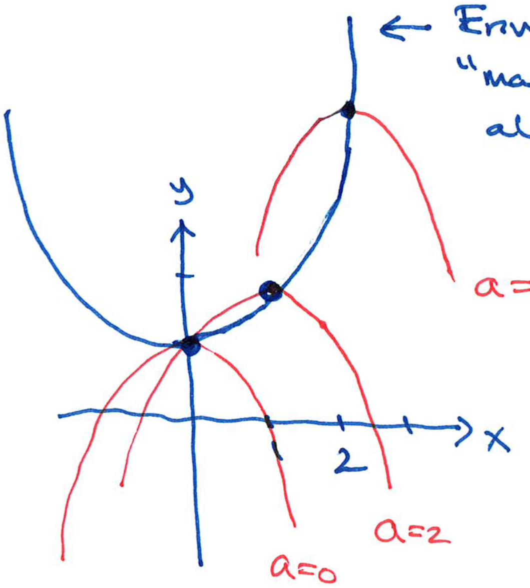
TA session: today 16-18 in D1-065

Exams: More orders $12 \times 6p + 4 \times 3p$

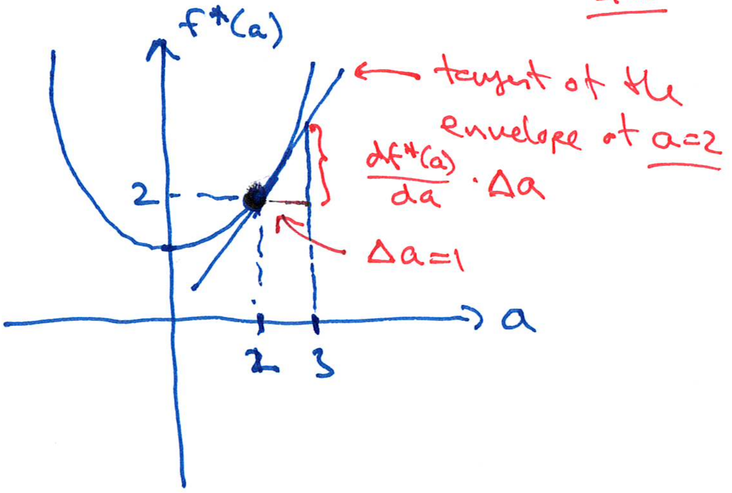
① Envelope Theorems

Ex: $a=2$
$\max f(x) = 1 + 2x - x^2$
$f'(x) = 2 - 2x = 0 \quad x = 1$
$f''(x) = -2 < 0 \Rightarrow f$ concave
$x^* = 1$ max. pt. $\quad f^* = f_{max} = f(1) = 2$ max value

$\Rightarrow \max f(x; a) = 1 + ax - x^2$
$f' = a - 2x = 0 \quad x = a/2$
$f'' = -2 < 0 \Rightarrow f$ concave for all a
$x^*(a) = a/2 \quad f^*(a) = 1 + a(a/2) - (a/2)^2 = 1 + a^2/4$



Ex: $a = 3 \quad f^*(3) = 1 + 9/4 = 3{,}25$



$a=2$: $x^*(2) = 1$ ← max pt when $a=2$
$f^*(2) = 2$ ← max value when $a=2$

Approximation:

$$f^*(3) \simeq \underbrace{f^*(2)}_{2} + \frac{df^*(a)}{da} \cdot \underbrace{\Delta a}_{3-2}$$

Envelope thm for the unconstrained case $\boxed{\max/\min f(\underline{x})}$

$$\boxed{\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))}$$

Ex: $f'_a = x$

$f'_a(x^*(a)) = x^*(a)$

At a=2: $\left.\frac{df^*(a)}{da}\right|_{a=2} = x^*(2) = \underline{1}$

$f^*(3) \simeq f^*(2) + \Delta a \cdot \frac{df^*(a)}{da}$

$= 2 + 1 \cdot 1 = \underline{\underline{3}}$

$$\boxed{\begin{array}{l}\text{Envelope thm in the Lagrange/Kuhn-Tucker case} \\ \dfrac{df^*(a)}{da} = \mathcal{L}'_a(\underline{x}^*(a); \lambda^*(a))\end{array}}$$

② Constrained optimization for quadratic functions

Ex I: $\min f(x,y,z) = x^2+2xy+4y^2+z^2$ when $x+y+z=1$

$\mathcal{L} = x^2+2xy+4y^2+z^2 - \lambda(x+y+z-1)$

$= \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 1)$

Reminder:

$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

$f'(\underline{x}) = 2A\underline{x} + B^T$

$H(f) = 2A$

$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

FOC: $\mathcal{L}'(\underline{x}) = \boxed{\begin{array}{r} 2A\underline{x} - \lambda(B^T) = \underline{0} \\ B\underline{x} = 1 \end{array}}$ $\Leftarrow \begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \\ \mathcal{L}'_z = 0 \end{cases}$

C:

Alt I: $2A\underline{x} = \lambda B^T$

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} x = A\underline{x} = \frac{\lambda}{2} \cdot B^T = \frac{\lambda}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$B\underline{x} = 1$

$(1 \; 1 \; 1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & \lambda/2 \\ 1 & 4 & 0 & \lambda/2 \\ 0 & 0 & 1 & \lambda/2 \\ 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow{-1} \; \xrightarrow{-1}$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & \lambda/2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & \lambda/2 \\ 0 & 0 & 1 & 1-\lambda/2 \end{array}\right) \xrightarrow{-1}$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & \lambda/2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & \lambda/2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1-\lambda} \end{array}\right)$$

$\lambda = 1 \quad z = 1/2 \quad y = 0 \quad x = 1/2$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \lambda \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1/2 & 0 \\ 1 & 4 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right) \end{array}$$

Conclusion:

$(x, y, z; \lambda) = (1/2, 0, 1/2; 1)$

SOC: $h = x^T A x - 1 \cdot (Bx - 1)$

convex?

$H(h) = 2A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $D_1 = 2$, $D_2 = 12$, $D_3 = 24$

pos. defn.

h convex $\Rightarrow$ SOC $f_{min} = f(1/2, 0, 1/2) = 1/2$

Alt 2: $2A\underline{x} = \lambda B^T$

$A\underline{x} = \frac{\lambda}{2} B^T$

$|A| \neq 0$: $\underline{x} = A^{-1} \cdot (\frac{\lambda}{2} B^T) = \frac{\lambda}{2} A^{-1} B^T$

$B\underline{x} = 1$

$B \cdot (\frac{\lambda}{2} A^{-1} B^T) = 1$

$\frac{\lambda}{2} \cdot B A^{-1} B^T = 1$

$\lambda \cdot (B A^{-1} B^T) = 2 \Rightarrow \lambda = \frac{2}{BA^{-1}B^T}$

$\underline{x} = \frac{1}{BA^{-1}B^T} \cdot (A^{-1}B^T)$

pos.

Claim: A pos. defn. $\Downarrow$ A^{-1} pos. defn.

A pos. defn, $\lambda_1, \ldots, \lambda_n > 0$

$\Rightarrow \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n} > 0$ are the eigenvalues of A^{-1} $\Rightarrow A^{-1}$ pos. defn.

$A\underline{x} = \lambda \underline{x} \Rightarrow A^{-1}A\underline{x} = A^{-1}\lambda\underline{x}$

$\underline{x} = \lambda A^{-1}\underline{x} \Rightarrow \frac{1}{\lambda}\underline{x} = A^{-1}\underline{x}$

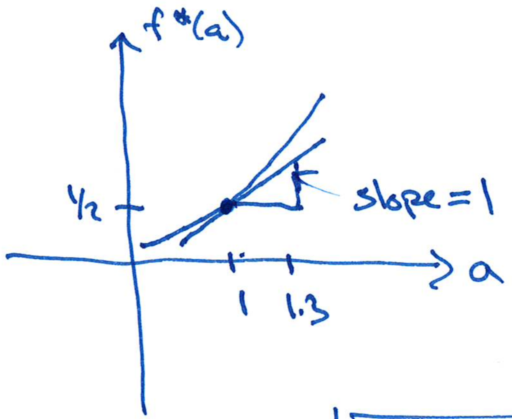
We have found:

min $f = x^2+2xy+4y^2+z^2$ when $x+y+z=1$ $\Rightarrow$ $x^*(1) = 1/2$, $y^*(1) = 0$, $z^*(1) = 1/2$, $\lambda^*(1) = 1$ } min pt

$\underline{f^*(1) = 1/2}$ } min value

What if we change the constraint to $x+y+z=1.3$

$$\Rightarrow \boxed{\min f = \underline{x}^T A \underline{x} \text{ when } x+y+z=a}$$



$$f^*(1.3) \simeq f^*(1) + \Delta a \cdot \frac{df^*(a)}{da}$$

$f^*(1) = 1/2$, $\Delta a = 1.3-1 = 0.3$, $\frac{df^*(a)}{da} = \lambda^*(1) = 1$

$$= \underline{\underline{0.8}}$$

Envelope thm: $\boxed{\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \lambda^*(a))}$

Ex: $L = \underline{x}^T A \underline{x} - \lambda(\overbrace{x+y+z}^{B\underline{x}} - a) = \ldots + \lambda a$

$L'_a = \lambda \underset{\text{Env.Th.}}{\Rightarrow} \frac{df^*(a)}{da} = L'_a(x^*(a); \lambda^*(a)) = \lambda^*(a)$

KT, std form

Ex 2: $\boxed{\max f(x,y,z) = x^2+2xy+4y^2+z^2 \text{ when } x^2+y^2+z^2 \le 2}$

$\max f(\underline{x}) = \underline{x}^T A \underline{x}$ when $\underline{x}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{x} \le 2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x}^T I \underline{x}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x}^T \underline{x}$

$L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T I \underline{x} - 2)$

FOC: $L'(\underline{x}) = 2A\underline{x} - \lambda(2I\underline{x}) = \underline{0}$

C: $\underline{x}^T I \underline{x} < 2$ or $\underline{x}^T I \underline{x} = 2$

CSC: $\lambda = 0$ $\qquad \lambda \ge 0$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D_1 = 1$, $D_2 = 3$, $D_3 = 3$

pos. def.

$|A| = D_3 = 3 \neq 0$ so A^{-1} exists

NB: $\lambda = 0 \Rightarrow 2A\underline{x} = \underline{0}$

$A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0}$

$\Rightarrow \underline{(0,0,0;0)}$ $f = 0$

B: $2A\underline{x} - \lambda\cdot(2I\underline{x}) = 0 \;|:2$ & $\underline{x}^T I \underline{x} = 2$ & $\lambda \ge 0$

$A\underline{x} - \lambda I \underline{x} = \underline{0}$ $\qquad \underline{x}^T \underline{x} = 2$ $\qquad$ ok

$(A - \lambda I)\underline{x} = \underline{0}$ $\qquad \|\underline{x}\|^2 = 2$

$\underline{x} = \underline{0}$ or $|A - \lambda I| = 0$ $\qquad \|x\| = \sqrt{2}$

~~$\underline{x}^T I \underline{x} = 0 \neq 2$~~

$\boxed{\underline{x} \text{ is an eigenvector of } A \text{ with eigenval. } \lambda}$

$(x_1\ x_2\ x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$= x_1^2 + x_2^2 + x_3^2$

$= \|x\|^2$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

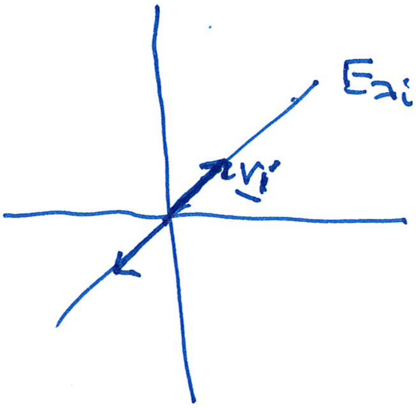
$$+ (1-\lambda) \cdot \begin{vmatrix} 1-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot(\lambda^2 - 5\lambda + 3) = 0$$

$$\underline{\lambda_1 = 1} \quad \text{or} \quad \lambda = \frac{5 \pm \sqrt{25 - 4\cdot 3}}{2}$$

$$= \frac{1}{2}(5 \pm \sqrt{13})$$

$$\underline{\lambda_2 = \tfrac{1}{2}(5+\sqrt{13})} \quad \underline{\lambda_3 = \tfrac{1}{2}(5-\sqrt{13})}$$



$$\frac{1}{2}(5+\sqrt{13}) > 1 > \frac{1}{2}(5-\sqrt{13})$$

$$E_1 = \text{span}(\underline{v_1}) \quad E_{\lambda_2} = \text{span}(\underline{v_2})$$

$$E_{\lambda_3} = \text{span}(\underline{v_3})$$

Candidate pts:

$$\underline{x} = \pm \frac{\underline{v_1}}{\|\underline{v_1}\|} \cdot \sqrt{2} \qquad \lambda_1 = 1 \quad \text{etc}$$

$$\Downarrow$$

Ax = λx since x is eigenvalue

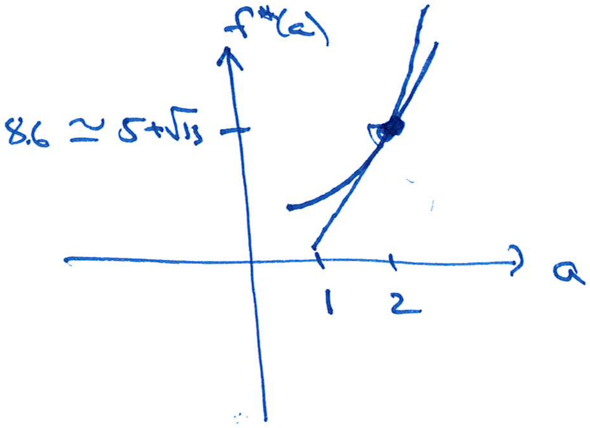
$$\left(\pm\sqrt{2}\cdot\frac{\underline{v_1}}{\|\underline{v_1}\|} ; 1\right) \qquad f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T \cdot \lambda_1 \cdot \underline{x} = \lambda_1 \underline{x}^T \underline{x} = 2\lambda_1 = \underline{\underline{2}}$$

$$\left(\pm\sqrt{2}\,\frac{\underline{v_2}}{\|\underline{v_2}\|} ; \tfrac{1}{2}(5+\sqrt{13})\right) \qquad f(\underline{x}) = 2\lambda_2 = \underline{\underline{5+\sqrt{13}}}$$

$$\left(\pm\sqrt{2}\,\frac{\underline{v_3}}{\|\underline{v_3}\|} ; \tfrac{1}{2}(5-\sqrt{13})\right) \qquad f(\underline{x}) = 2\lambda_3 = \underline{\underline{5-\sqrt{13}}}$$

$$f_{max} = \underline{\underline{5+\sqrt{13}}}$$

What about: $\max\ x^2+2xy+4y^2+z^2$ wh $x^2+y^2+z^2 \le \underbrace{1}_{a}$



$\underline{a=2:}\ f^*(2) = 5+\sqrt{13} \simeq \underline{\underline{8.6}}$

$$f^*(1) \simeq f^*(2) + \Delta a \cdot \frac{df^*(a)}{da}$$

$$f^*(2) = 5+\sqrt{13} \simeq 8.6, \qquad \Delta a = 1-2 = -1, \qquad \frac{df^*(a)}{da} = \lambda^*(2) = \frac{5+\sqrt{13}}{2}$$

$$\simeq 8.6 - 4.3 = \underline{\underline{4.3}}$$

$$L = \underline{x}^T A \underline{x} - \lambda(x^2+y^2+z^2-a)$$

$$L'_a = \lambda \Rightarrow \frac{df^*(a)}{da} = L'_a(\underline{x}^*(a);\lambda^*(a)) = \lambda^*(a)$$

What about $\boxed{\max\ 2x^2+2xy+4y^2+z^2 \text{ wh } x^2+y^2+z^2 \le 2}$

Look at: $\max\ bx^2+2xy+4y^2+z^2$ wh $x^2+y^2+z^2 \le 2$

$\underline{b=1:}$

$$f^*(1) = 5+\sqrt{13}$$
$$\lambda^*(1) = \frac{5+\sqrt{13}}{2}$$
$$\underline{x}^*(1) = \pm\sqrt{2}\,\frac{\underline{v}_2}{\|\underline{v}_2\|}$$

$$L = bx^2+2xy+4y^2+z^2 - \lambda(x^2+y^2+z^2-2)$$

$$L'_b = x^2 \underset{\text{Env.Thm.}}{\Longrightarrow} \frac{df^*(b)}{db} = x^*(b)^2$$

Chnge b: $f^*(2) \simeq f^*(1) + \Delta b \cdot \dfrac{df^*(b)}{db}$

$$= (5+\sqrt{13}) + 1 \cdot x^*(b)^2$$

Ex: max $f(x,y,z) = 3x^2 + 2xy + 3y^2 + z^2$ when $x^2+y^2+z^2 \leq 2$

(Similar to Ex 2, but more explicit computations)

$= \underline{x}^T A \underline{x}$ $\qquad$ $\underline{x}^T I \underline{x}$

$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D_1 = 3$, $D_2 = 8$, $D_3 = 8$ $\qquad$ $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

pos. defn.

$\mathcal{L} = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T I \underline{x} - 2)$

FOC: $\mathcal{L}'(\underline{x}) = 2A\underline{x} - \lambda \cdot (2I\underline{x}) = \underline{0}$

C: $\boxed{\underline{x}^T I \underline{x} = 2}$ or $\boxed{\underline{x}^T I \underline{x} < 2}$

CSC: $\boxed{\lambda \geq 0}$ $\qquad$ $\boxed{\lambda = 0}$

B $\qquad$ NB

NB case: $\lambda = 0 \Rightarrow 2A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0}$ since A pos defn.

$\Rightarrow$ Cand. pt: $\underline{(0,0,0;0)}$ $f = 0$

B case: (FOC) $2A\underline{x} - 2\lambda I \underline{x} = \underline{0}$

$A\underline{x} - \lambda I \underline{x} = \underline{0}$

$(A - \lambda I)\underline{x} = \underline{0}$

$\underline{x} = \underline{0}$ or $\underline{x}$ eigenvector with eigenvalue λ

$\underline{x}^T I \underline{x} = 0 \neq 2$

no cand. pts.

Eigenvalues of A: $\begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$

$\underline{\lambda = 1}$, $\underline{\lambda = 2}$, $\underline{\lambda = 4}$

Eigenvectors:

$\lambda=1$ $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\underline{v_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ base of E_1

$\lambda=2$: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ base of E_2

$\lambda=4$: $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ base of E_4

FOC: $\underline{x} = t\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\lambda=1$ or

$\underline{x} = t\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\lambda=2$ or

$\underline{x} = t\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\lambda=4$

CSC: $\lambda \geq 0$ ok in all cases

C: $\underline{x}^T I \underline{x} = \underline{x}^T\underline{x} = \|\underline{x}\|^2 = 2$

$\left\| t\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\| = t^2 = 2 \Rightarrow t = \pm\sqrt{2}$

$\left\| t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\| = 2t^2 = 2$ $t = \pm 1$

$\left\| t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\| = 2t^2 = 2$ $t = \pm 1$

$f(\underline{x}) = \underline{x}^T A \underline{x} = \underbrace{\underline{x}^T \lambda \underline{x}}_{\text{eigenvectors}}$
$= \lambda\cdot\underline{x}^T\underline{x} = \lambda\cdot 2 = \underline{2\lambda}$

Candidate pts:

$(0,0,\pm\sqrt{2};1)$ $f=2$

$(1,-1,0;2)$, $(-1,1,0;2)$ $f=4$

$(1,1,0;4)$, $(-1,-1,0;4)$ $f=8$

$\underline{\underline{f_{max}=8}}$ at $\underline{(1,1,0)}$, $\underline{(-1,-1,0)}$ with $\underline{\lambda=4}$ since $x^2+y^2+z^2 \leq 2$ is bounded and NDCQ holds at all adm. pts.