Note: Optimal portfolio theory and minimal variance portfolios

We consider a situation where we are to construct a portfolio of securities, and n securities (numbered $1-n$) are available to us. Let R_i be the return of security i for $1 \le i \le n$, and assume:

$E(R_i) = \mu_i$ – expected return of security i is μ_i

$Var(R_i) = \sigma_i^2$ – std. dev. of the return of security i is σ_i, and the variance is σ_i^2

$Cov(R_i, R_j) = \sigma_{ij}$ – the covariance of the returns of securities i and j ~~is~~ σ_{ij} (with $\sigma_{ii} = \sigma_i^2$)

We ~~[illegible]~~ also assume that these parameters are known and constant.

We choose a portfolio with portfolio weights $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, where x_i is the weight of ~~portfolio~~ security i in the portfolio. We allow $x_i < 0$ (short selling). Let R be the return of the chosen portfolio. Then $R = x_1 R_1 + x_2 R_2 + \dots + x_n R_n$, and we have:

$$\boxed{\begin{aligned} E(R) &= x_1 E(R_1) + \dots + x_n \cdot E(R_n) \\ &= x_1 \mu_1 + \dots + x_n \mu_n = \underline{\mu}^T \cdot \underline{x} \\ Var(R) &= Cov(x_1 R_1 + \dots + x_n R_n, x_1 R_1 + \dots + x_n R_n) \\ &= x_1^2 Var(R_1) + x_1 x_2 Cov(R_1, R_2) + \dots \\ &\quad \dots + x_n^2 Var(R_n) = \underline{x}^T \Sigma \underline{x} \end{aligned}}$$

$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$ vector of expected returns

$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & & & \\ \sigma_{1n} & \sigma_{2n} & \dots & \sigma_n^2 \end{pmatrix}$ covariance matrix

Note: i) The portfolio weights satisfy $x_1 + x_2 + \dots + x_n = 1$, or $\underline{e}^T \cdot \underline{x} = 1$ with $\underline{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ the constant vector with value 1 in each component.

ii) Since $Var(R) \ge 0$ for any portfolio weights $\underline{x}$, we have that Σ is positive semidefinite.

Additional assumptions and their interpretation

i) Σ is positive definite – if this is not the case, then it is positive semidefinite with at least one eigenvalue equal to 0, and this means

$$\Sigma \underline{v} = 0 \Rightarrow \underline{v}^T \Sigma \underline{v} = 0$$

for some non-zero $\underline{v}$ (and we may assume that $v_1 + v_2 + \dots + v_n = 1$ (if not, divide by the sum of $\underline{v}$'s components). This implies that the portfolio $\underline{v}$ has constant return, and therefore one of the securities may be obtained as a linear combination of the others

ii) $\underline{\mu}$ and $\underline{e}$ are linearly independent vectors – if not, $\underline{\mu}$ is a scalar multiple of $\underline{e}$, a constant vector, and therefore all securities have the same return

We conclude that assumption (1) and (2) are reasonable, and we shall assume that they are satisfied.

Problem A: $\min \underbrace{\underline{x}^T \Sigma \underline{x}}_{Var(R)}$ when $\underline{e}^T \underline{x} = 1$ – minimal variance portfolio (without considering the portfolio return)

Problem B: $\min \underline{x}^T \Sigma \underline{x}$ when $\begin{cases} \underline{e}^T \underline{x} = 1 \\ \underline{\mu}^T \underline{x} = r \end{cases}$ – minimal variance portfolio among portfolios with expected return r

We can solve Problem A-B and find a unique solution, assuming that condition i) - ii) hold.

Problem A: $\min \ \underline{x}^T \Sigma \underline{x}$ wrt $\underline{e}^T \underline{x} = 1$ Lagrange problem

$$L = \underline{x}^T \Sigma \underline{x} - \lambda(\underline{e}^T \underline{x} - 1)$$

FOC: $L'(\underline{x}) = 2\Sigma \underline{x} - \lambda(\underline{e}^T)^T = \boxed{\begin{array}{l} 2\Sigma \underline{x} - \lambda \underline{e} = \underline{0} \\ \underline{e}^T \underline{x} = 1 \end{array}}$ Lagrange conditions

C:

FOC: $2\Sigma \underline{x} = \lambda \underline{e}$

$$\Sigma \underline{x} = \frac{\lambda}{2} \underline{e}$$

$$\underline{x} = \Sigma^{-1}\left(\frac{\lambda}{2}\underline{e}\right) = \frac{\lambda}{2}\Sigma^{-1}\underline{e}$$

Σ positive definite $\Rightarrow \lambda_1, \dots, \lambda_n > 0 \Rightarrow |\Sigma| = \lambda_1 \lambda_2 \cdots \lambda_n > 0$ $\Rightarrow \Sigma^{-1}$ exists (and is positive defn. since its eigenvalues are $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n} > 0$)

C: $\underline{e}^T \underline{x} = \underline{e}^T\left(\frac{\lambda}{2}\Sigma^{-1}\underline{e}\right) = \frac{\lambda}{2}\underline{e}^T \Sigma^{-1} \underline{e} = 1 \quad | \cdot 2$

$$\lambda \cdot \left(\underline{e}^T \Sigma^{-1} \underline{e}\right) = 2 \qquad \lambda = \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}} > 0$$

since Σ^{-1} is positive defn, $\underline{e}^T \Sigma^{-1} \underline{e} > 0$

Candidate pt: $\underline{x}^* = \frac{\lambda}{2}\Sigma^{-1}\underline{e} = \frac{1}{\underline{e}^T \Sigma^{-1} \underline{e}} \cdot (\Sigma^{-1}\underline{e})$

$$\lambda^* = \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}}$$

SOC: $h = L(\underline{x}; \lambda^*) = \underline{x}^T \Sigma \underline{x} - \underbrace{\frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}}}_{\text{pos. number}} \cdot \underbrace{(\underline{e}^T \underline{x} - 1)}_{\text{linear form}}$

keep $\underline{x}$ as variables, substitute $\lambda = \lambda^*$

$H(h) = 2\Sigma$ pos. defn. since Σ pos. defn. $\Rightarrow h$ convex

$\Rightarrow (\underline{x}^*; \lambda^*) = \left(\frac{1}{\underline{e}^T \Sigma^{-1} \underline{e}} \cdot \Sigma^{-1}\underline{e}; \ \frac{2}{\underline{e}^T \Sigma^{-1} \underline{e}}\right)$ is minimum in Prob A.

$$\boxed{\text{Minimum Variance Portfolio: } \underline{x} = \frac{1}{\underline{e}^T \Sigma^{-1} \underline{e}} \cdot \Sigma^{-1}\underline{e}}$$

<u>Problem B:</u> $\min \ \underline{x}^T \Sigma \underline{x}$ when $\begin{cases} \underline{e}^T \underline{x} = 1 \\ \underline{\mu}^T \underline{x} = r \end{cases}$ Lagrange problem

$$L = \underline{x}^T \Sigma \underline{x} - \lambda_1 (\underline{e}^T \underline{x} - 1) - \lambda_2 (\underline{\mu}^T \underline{x} - r)$$

<u>FOC</u>: $L'(\underline{x}) =$ $\boxed{\begin{aligned} 2\Sigma \underline{x} - \lambda_1 (\underline{e}) - \lambda_2 \underline{\mu} &= \underline{0} \\ \underline{e}^T \cdot \underline{x} &= 1 \\ \underline{\mu}^T \cdot \underline{x} &= r \end{aligned}}$ Lagrange conditions

<u>C</u>:

<u>FOC</u>: $2\Sigma \underline{x} = \lambda_1 \underline{e} + \lambda_2 \underline{\mu} \Rightarrow \Sigma \underline{x} = \frac{\lambda_1}{2} \underline{e} + \frac{\lambda_2}{2} \underline{\mu} \Rightarrow \underline{x} = \Sigma^{-1} \left(\frac{\lambda_1}{2} \underline{e} + \frac{\lambda_2}{2} \underline{\mu} \right)$

$$= \frac{\lambda_1}{2} \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu}$$

<u>C</u>: $\underline{e}^T \underline{x} = \underline{e}^T \left(\frac{\lambda_1}{2} \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu} \right) = \frac{\lambda_1}{2} \underline{e}^T \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \underline{e}^T \Sigma^{-1} \underline{\mu} = 1 \quad | \cdot 2$

$$\Rightarrow \boxed{\lambda_1 \cdot (\underline{e}^T \Sigma^{-1} \underline{e}) + \lambda_2 (\underline{e}^T \Sigma^{-1} \underline{\mu}) = 2}$$

positive numbers since Σ^{-1} pos. defin.

$\underline{\mu}^T \underline{x} = \underline{\mu}^T \left(\frac{\lambda_1}{2} \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu} \right) = \frac{\lambda_1}{2} \underline{\mu}^T \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \underline{\mu}^T \Sigma^{-1} \underline{\mu} = r \quad | \cdot 2$

$$\Rightarrow \boxed{\lambda_1 (\underline{\mu}^T \Sigma^{-1} \underline{e}) + \lambda_2 (\underline{\mu}^T \Sigma^{-1} \underline{\mu}) = 2r}$$

$\Rightarrow$ 2x2 lin. sys. to find λ_1, λ_2:
$$\begin{pmatrix} \underline{e}^T \Sigma^{-1} \underline{e} & \underline{e}^T \Sigma^{-1} \underline{\mu} \\ \underline{\mu}^T \Sigma^{-1} \underline{e} & \underline{\mu}^T \Sigma^{-1} \underline{\mu} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2r \end{pmatrix}$$

rewrite as:
$$(\underline{e} | \underline{\mu})^T \cdot \Sigma^{-1} \cdot (\underline{e} | \underline{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2r \end{pmatrix}$$

$$B \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2r \end{pmatrix} \quad \text{with} \quad B = (\underline{e} | \underline{\mu})^T \Sigma^{-1} (\underline{e} | \underline{\mu})$$

<u>Note:</u> B positive definite 2x2 matrix

i) B pos. defin. $\Leftrightarrow$ B pos. semidefin. and $|B| \neq 0$

ii) B pos. semidefin: $(\lambda_1 \ \lambda_2) B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \underline{\lambda}^T B \underline{\lambda} = (\lambda_1 \underline{e} + \lambda_2 \underline{\mu})^T \Sigma^{-1} (\lambda_1 \underline{e} + \lambda_2 \underline{\mu}) \geq 0$ for all λ_1, λ_2 since Σ^{-1} is positive defin.

iii) <u>B pos. defn:</u> Assume that there are $\binom{\lambda_1}{\lambda_2} \neq \binom{0}{0}$ such that

$$\underline{\lambda}^T B \underline{\lambda} = 0 = (\lambda_1 \lambda_2)(\underline{e}|\underline{\mu})^T \Sigma^{-1} (\underline{e}|\underline{\mu}) \binom{\lambda_1}{\lambda_2} = 0$$

$$\Rightarrow (\lambda_1 \underline{e} + \lambda_2 \underline{\mu})^T \cdot \Sigma^{-1} (\lambda_1 \underline{e} + \lambda_2 \underline{\mu}) = 0$$

Σ^{-1} is pos. defn.

$$\Rightarrow \lambda_1 \underline{e} + \lambda_2 \underline{\mu} = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$$

$\underline{e}, \underline{\mu}$ lin. independent

This is a contradiction, therefore $\underline{\lambda}^T B \underline{\lambda} > 0$ for all $\underline{\lambda} \neq \underline{0}$, and B is pos. defn.

$$B \cdot \binom{\lambda_1}{\lambda_2} = \binom{2}{2r} \Rightarrow \binom{\lambda_1}{\lambda_2} = B^{-1} \cdot \binom{2}{2r}$$

unique solution for λ_1, λ_2 since $|B| \neq 0$ and B^{-1} exists.

<u>Candidate pt:</u>

$$\underline{x}^* = \frac{\lambda_1}{2} \Sigma^{-1} \underline{e} + \frac{\lambda_2}{2} \Sigma^{-1} \underline{\mu} = \frac{1}{2} \Sigma^{-1} (\lambda_1 \underline{e} + \lambda_2 \underline{\mu}) = \frac{1}{2} \Sigma^{-1} (\underline{e}|\underline{\mu}) \underline{\lambda}$$

$$= \frac{1}{2} \Sigma^{-1} (\underline{e}|\underline{\mu}) B^{-1} \binom{2}{2r} = \underline{\Sigma^{-1} (\underline{e}|\underline{\mu}) \left[(\underline{e}|\underline{\mu})^T \Sigma^{-1} (\underline{e}|\underline{\mu}) \right]^{-1} \cdot \binom{1}{r}}$$

$$\underline{\lambda}^* = B^{-1} \cdot \binom{2}{r} = \underline{\left[(\underline{e}|\underline{\mu})^T \Sigma^{-1} (\underline{e}|\underline{\mu}) \right]^{-1} \cdot \binom{2}{2r}}$$

<u>SOC:</u>

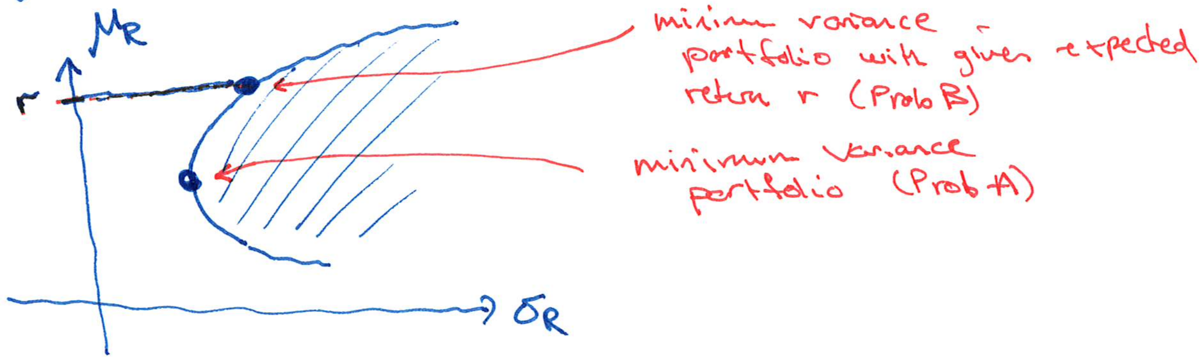
$$h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*) = \underline{x}^T \Sigma \underline{x} - \underbrace{\lambda_1^* (\underline{e}^T \underline{x} - 1) - \lambda_2^* (\underline{\mu}^T \underline{x} - r)}_{\text{linear in } \underline{x}}$$

$$\Rightarrow H(h) = 2\Sigma \Rightarrow h \text{ convex} \xrightarrow[\text{SOC}]{} (\underline{x}^*; \underline{\lambda}^*) \text{ is } \underline{\text{minimum}} \text{ in Prob B}$$

pos. defn.

<u>Picture:</u>

Let $\sigma_R = \sqrt{Var(R)}$. Then all possible values of (μ_R, σ_R) are given geometrically as



Ex: $n=3$ $\quad \underline{\mu} = \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix}$ $\quad \Sigma = \begin{pmatrix} 0.1000^2 & 0.0018 & 0.0011 \\ 0.0018 & 0.1044^2 & 0.0026 \\ 0.0011 & 0.0026 & 0.1411^2 \end{pmatrix}$

vector of expected returns (not constant)

Covariance matrix (pos. defn.)

Prob A: Minimum variance portfolio

$$\underline{x}^* = \frac{1}{\underline{e}^T \Sigma^{-1} \underline{e}} \cdot \Sigma^{-1} \underline{e}$$

$$= \frac{1}{189.234} \begin{pmatrix} 83.5148 \\ 69.2237 \\ 36.5906 \end{pmatrix} = \begin{pmatrix} 0.441 \\ 0.366 \\ 0.193 \end{pmatrix}$$

Compute: With some help, for instance python code in [EJ], p.174:

$$\Sigma^{-1} = \begin{pmatrix} 103.312 & -16.2035 & -3.59367 \\ -16.2035 & 97.2356 & -11.8085 \\ -3.59367 & -11.8085 & 51.9997 \end{pmatrix}$$

$$\underline{e}^T \Sigma^{-1} \underline{e} = (1\ 1\ 1)\Sigma^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 189.329$$

$$\Sigma^{-1} \underline{e} = \begin{pmatrix} 83.5148 \\ 69.2237 \\ 36.5906 \end{pmatrix}$$

Portfolio: 44.1% in Sec #1
36.6% in Sec #2
19.3% in Sec #3