	Plan	
1	Key Problems:	2.1bdf, 2.2, 2.4, 3.3bc, 3.4bc, 4.1def, 4.2d, 4.3, 4.4b, 4.5b
2	Textbook Problems:	2.7, 2.14
3	Exam Problems:	Midterm 10/2022 Q7

} No time for these problems

① Key problems

2.1

$$A = (\underline{v_1}|\underline{v_2}|\dots|\underline{v_5}) = \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 3 & 3 & 3 & 4 & 2 \\ 4 & 4 & 4 & 5 & 3 \end{pmatrix} \begin{matrix} \\ \leftarrow -3 \\ \leftarrow -4 \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 0 & 6 & -12 & -14 & -10 \\ 0 & 8 & -16 & -19 & -13 \end{pmatrix} \leftarrow -4/3 \rightarrow \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 0 & 6 & -12 & -14 & -10 \\ 0 & 0 & 0 & 1/3 & 1/3 \end{pmatrix} \cdot 3$$

$$-19 + \frac{14 \cdot 4}{3} = \frac{-57+56}{3}$$

$$-13 + \frac{40}{3} = \frac{-39+40}{3}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 0 & 6 & -12 & -14 & -10 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned} x_1 - x_2 + 5x_3 + 6x_4 &= 4 \\ 6x_2 - 12x_3 - 14x_4 &= -10 \\ -x_4 &= 1 \end{aligned}$$

b) $x_1\underline{v_1} + x_2\underline{v_2} + x_3\underline{v_3} + x_4\underline{v_4} = \underline{v_5}$?

Yes.

x_3 free int. many solutions

f) $x_3 = 0$: $6\underline{v_1} - 4\underline{v_2} - \underline{v_4} = \underline{v_5}$

$x_3 = 2$: $2\underline{v_3} - \underline{v_4} = \underline{v_5}$

$x_4 = -1$

$6x_2 = 12x_3 + 14(-1) - 10$

$= 12x_3 - 24$

$x_2 = 2x_3 - 4$

d) $(\underline{v_1}|\underline{v_2}|\underline{v_5}) \rightarrow \begin{pmatrix} 1 & -1 & 4 \\ 0 & 6 & -10 \\ 0 & 0 & 1 \end{pmatrix}$

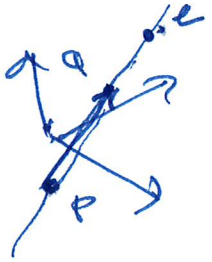
$x_1 = (2x_3 - 4) - 5x_3 - 6(-1) + 4$

$-3x_3 + 6$

$\Rightarrow \underline{v_1}, \underline{v_2}, \underline{v_5}$ are lin. independent

2.2. $P = (1,2,1)$

$Q = (4,5,3)$



$$\underline{v} = \vec{PQ} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \begin{matrix} \leftarrow 4-1 \\ \leftarrow 5-2 \\ \leftarrow 3-1 \end{matrix}$$

$$x = 1+3t$$
$$y = 2+3t$$
$$z = 1+2t$$

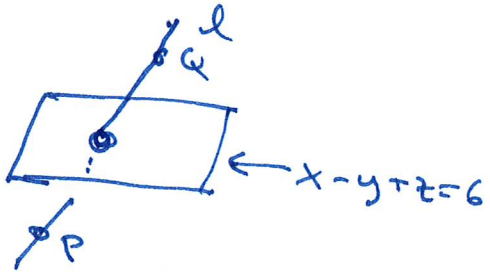
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{OP} + t\vec{PQ} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+3t \\ 2+3t \\ 1+2t \end{pmatrix}$$

Intersection with $\underline{x-y+z=6}$: $(1+3t)-(2+3t)+(1+2t)=6$

$2t = 6$

$t = 3$

$(x,y,z) = (10,11,7)$



$\text{span}(\underline{v}_1,\dots,\underline{v}_8)$

2.4.

A 8×8 matrix

$\text{rk}\, A = 7$

a) $\dim \text{Col}(A) = \text{rk}(A) = \underline{7}$

$\dim \text{Null}(A) = n - \text{rk}(A) = 8-7 = \underline{1}$

" all soln's of $A\underline{x} = \underline{0}$

b) Soln's of $A\underline{x} = \underline{0}$ = Number of vectors in Null(A)

= Inf. many (one degree of freed.)

c) Soln's of $A\underline{x} = \underline{b}$:

$$\left(\begin{array}{c|c} A & \underline{b} \end{array}\right) \to \dots \to \left(\begin{array}{ccc|c} \circledast & \cdots & * & * \\ & \ddots & & \vdots \\ 0 & & \circledast & \vdots \\ 0 & \cdots & 0 & ? \end{array}\right)$$

$? = 0$: Inf. many soln's

$? \neq 0$: no soln's

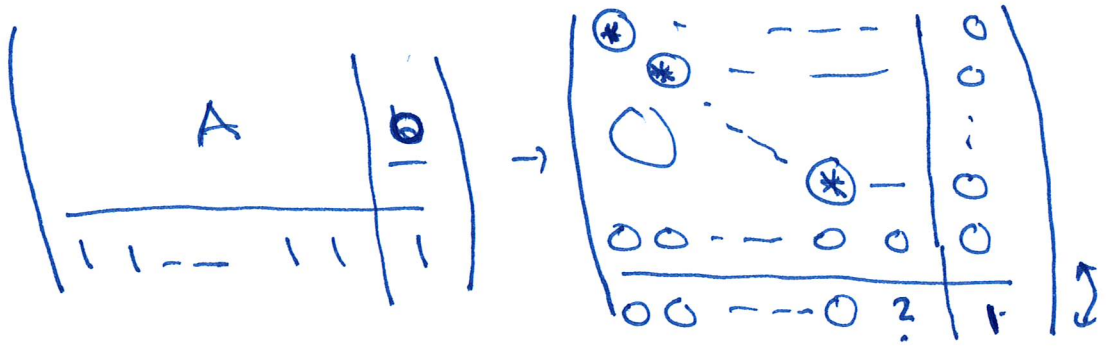
$? = 0 \iff \underline{b}$ is in Col(A).

d) No of sol'n of

$A\underline{x} = \underline{0}$

and

$x_1 + x_2 + \ldots + x_8 = 1$



$? = 0$: no solutions

$? \neq 0$: one unique solution

3.3.

b) $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & 7 & 3 \\ 4 & 5 & 11 & 10 \end{pmatrix}$

$M_{123,123} = 1(-46) - 3(-6) + 2\cdot(14)$

$= -46 + 18 + 28 = 0$

$M_{12,12} = -1 - 6 = -7 \neq 0$

$M_{123,124} = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 4 & 5 & 10 \end{vmatrix} = 1(-25) - 3(8) + 4\cdot 14$

$= -25 - 24 + 56 = 7 \neq 0$

$\text{rk } A = 3$

Base of $\text{Col}A = v_1, v_2, v_4$

c) $A = \begin{pmatrix} 1 & 4 & -3 & 1 \\ 2 & 7 & 1 & 2 \\ 1 & 3 & 4 & 1 \end{pmatrix}$

$M_{123,123} = 1(25) - 4(7) - 3(-1)$

$= 25 - 27 + 3 = 0$

$M_{123,124} = 0$

A matrix with two equal rows/cols have det = 0.

$\text{rk } A = 2$

Base of $\text{Col}(A)$: v_1, v_2

3.4. b) $A = \begin{pmatrix} 1 & 3 & 2 & -1 \\ 5 & 3 & 0 & 1 \\ 4 & 6 & 2 & 0 \end{pmatrix}$

$M_{123,123} = 2(6s - 12) + 2(3 - 3s)$

$= 12s - 24 + 6 - 6s = 6s - 18$

$= 6(s - 3)$

$s = 3$: $\begin{pmatrix} 1 & 3 & 2 & -1 \\ 3 & 3 & 0 & 1 \\ 4 & 6 & 2 & 0 \end{pmatrix}$ $\begin{matrix} \\ -3 \\ -4 \end{matrix}$

$\to \begin{pmatrix} 1 & 3 & 2 & -1 \\ 0 & -6 & -6 & 4 \\ 0 & -6 & -6 & 4 \end{pmatrix}$ $\begin{matrix} \\ \\ -1 \end{matrix}$

$\to \begin{pmatrix} 1 & 3 & 2 & -1 \\ 0 & 6 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\text{rk } A = 2$

$s \neq 3$: $M_{123,123} \neq 0$ $\text{rk } A = 3$

Conclusion:

$$\text{rk } A = \begin{cases} 3, & s \neq 3 \\ 2, & s = 3 \end{cases}$$

c) $A = \begin{pmatrix} 1 & a & b \\ a & b & c \end{pmatrix}$

$M_{12,12} = b - a^2 = 0$

~~$M_{12,23} = ac - b^2 = 0$~~

$M_{12,13} = c - ab = 0$

$\Leftrightarrow \operatorname{rk} A < 2$

$\operatorname{rk} A < 2 \Leftrightarrow$ 2-minors are zero

$b = a^2$, $c - a \cdot (a^2) = 0$

$c = a^3$

$ac - b^2 = a(a^3) - (a^2)^2 = 0$

Concl: $\operatorname{rk} A = \begin{cases} 1, & b = a^2 \text{ and } c = a^3 \\ 2, & \text{otherwise} \end{cases}$

4.1 d) $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

4.2 Yes.

Eigenvalues:

$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$+ (4-\lambda) \cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda = 4$ or $\lambda^2 - 6\lambda + 8 = 0$

$\lambda = 2$ or $\lambda = 4$

Eigenvectors:

$\lambda = 4$ (mult 2)

$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ + & 0 & - \end{pmatrix}$ $-x + z = 0$, y, z free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (v_1, v_2)

Base of E_4: $\{v_1, v_2\}$

$\lambda = 2$ (mult 1)

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ + & 0 & - \end{pmatrix}$ $x + z = 0$, $2y = 0$, z free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ (v_3)

Base of E_2: v_3

$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

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e) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$-\lambda(\lambda^2-1) - 1(-\lambda-1) + 1\cdot(1+\lambda) = 0$

$-\lambda(\lambda-1)(\lambda+1) + (\lambda+1) + (\lambda+1) = 0$

$(\lambda+1)(-\lambda(\lambda-1)+2) = 0$

$-\lambda^3+3\lambda+2 = (\lambda+1)(-\lambda^2+\lambda+2) = 0$

if $-\lambda^3+3\lambda+2=0$ has an integer solution, then it must divide 2

$\lambda = -1$ or $-\lambda^2+\lambda+2 = 0$

$\lambda^2-\lambda-2 = 0$

$(\lambda-2)(\lambda+1) = 0$

$\lambda = 2, \lambda = -1$

$\lambda = -1$ (mult. 2), $\lambda = 2$ (mult. 1)

f) $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{vmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$(0-\lambda)(0-\lambda)(0-\lambda) = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 0$

(entries on the diagonal if A is upper triangular)

$\lambda = 0$ (mult 3)

E_0: $\begin{pmatrix} 0 & \textcircled{1} & 1 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{pmatrix}$ x free, $y+z=0$, $z=0$

$\Rightarrow \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

4.3

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

A symmetric $\Rightarrow$ A is diagonalizable

$\Rightarrow$ A has 4 eigenvalues

A diagonalizable (nxn) $\Leftrightarrow$

i) A has n eigenval. and

ii) A has n lin.indep. eigenvectors

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 4 \\ 0 & 2-\lambda & 3 & 0 \\ 0 & 3 & 2-\lambda & 0 \\ 4 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} - 4\cdot\begin{vmatrix} 0 & 2-\lambda & 3 \\ 0 & 3 & 2-\lambda \\ 4 & 0 & 0 \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\left[+(1-\lambda)\cdot\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix}\right] - 4\cdot\left[4\cdot\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix}\right] = 0$$

$$(1-\lambda)^2(\lambda^2-4\lambda-5) - 4^2(\lambda^2-4\lambda-5) = 0$$

$$(\lambda^2-4\lambda-5)(\lambda^2-2\lambda+1-16) = 0$$

$\lambda^2-4\lambda-5=0$ or $\lambda^2-2\lambda-15=0$

$(\lambda-5)(\lambda+1)=0$ $\quad$ $(\lambda-5)(\lambda+3)=0$

$\lambda=5$, $\lambda=-1$, $\lambda=5$, $\lambda=-3$

4.4 b)

$$A = \begin{pmatrix} 0.77 & 0.46 \\ 0.23 & 0.54 \end{pmatrix}$$

Eigenvalues:

$$\lambda^2 - 1.31\lambda + 0.31 = 0$$

$$0.77 \cdot 0.54 - 0.23 \cdot 0.46$$

$$(\lambda - 1)(\lambda - 0.31) = 0$$

$\lambda = 1$ or $\lambda = 0.31$

Eigenvectors:

$\lambda = 1$: $\begin{pmatrix} -0.23 & 0.46 \\ 0.23 & -0.46 \end{pmatrix}$

$-0.23x + 0.46y = 0$ $\quad x = \frac{0.46y}{0.23} = 2y$

y free

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda = 0.31$: $\begin{pmatrix} 0.46 & 0.46 \\ 0.23 & 0.23 \end{pmatrix}$

$0.46x + 0.46y = 0$ $\quad x = -y$

y free

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.31 \end{pmatrix} \qquad P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$P^{-1}AP = D$

$A = PDP^{-1}$

$A^m = PD^mP^{-1}$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1^m & 0 \\ 0 & (+0.31)^m \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$\to 0$ when $m \to \infty$

$$\to \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} //$$

4.5

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}: \qquad \begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(\lambda^2-(e+i)\lambda + ei - fh) - b(d(i-\lambda)-fg) + c(dh-g(e-\lambda)) = 0$$

$$a\lambda^2 - \lambda^3 - a(e+i)\lambda + (e+i)\lambda^2 + a(ei-fh) - \lambda(ei-fh) - bdi + bd\lambda + bfg + cdh - cge + cg\lambda = 0$$

$$-\lambda^3 + (a+e+i)\lambda^2 - \lambda(a(e+i) + ei + bd + cg - fh) + a(ei-fh) - b(di-fg) + c(dh-eg) = 0$$

$$\boxed{-\lambda^3 + \mathrm{tr}(A)\lambda^2 - (M_{12,12} + M_{13,13} + M_{23,23})\lambda + \det(A) = 0}$$

$\mathrm{tr}(A) = a+e+i$

$M_{12,12} + M_{13,13} + M_{23,23} = (ae-bd) + (ai-cg) + (ei-fh)$

$\det(A) = a(ei-fh) - b(di-fg) + c(dh-eg)$

b) $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$\mathrm{tr}(A) = 3+3+3 = 9$

$M_{12,12} = 9-4 = 5$

$M_{13,13} = 9-1 = 8$

$M_{23,23} = 9-4 = 5$

$5+8+5 = 18$

$\det(A) = 3(5) - 2(4) + 1(1) = 8$

$$-\lambda^3 + 9\lambda^2 - 18\lambda + 8 = 0$$

Key Problems

You should do the Key Problems as soon as possible after each lecture. They test your understanding of the most important theories and methods presented in the lecture. You can work on Key Problems and get help in the Exercise session/TA session the following Monday (Campus Bergen: Wednesday).

Problem 1.
Solve the linear system $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination:

a) $A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 1 & 8 \\ 4 & 5 & 9 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 11 \\ 15 \\ 23 \end{pmatrix}$

b) $A = \begin{pmatrix} 4 & 5 & 3 & 11 \\ 2 & 5 & 0 & 3 \\ 3 & 2 & 3 & 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 23 \\ 17 \\ 14 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 3 & 1 & 5 & 18 \\ 2 & 4 & 2 & 9 & 31 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 8 \\ 28 \\ 48 \end{pmatrix}$

d) $A = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 4 & 1 \\ 7 & 7 & 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 13 \\ 11 \\ 17 \end{pmatrix}$

Problem 2.
Find the rank of the following matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 3 & 0 & 2 & 3 & 1 \\ 3 & 6 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \end{pmatrix}$$

Problem 3.
Let V be the set of solutions of the linear system $A\mathbf{x} = \mathbf{b}$ from Problem 1(b). What is the dimension of V? Describe the set V geometrically.

Problem 4.
We consider the following linear system. Find all solutions that satisfies $x + w = y + z$:

$$\begin{array}{rcrcrcrcr} x & + & y & + & 2z & + & 4w & = & 6 \\ x & + & 2y & + & 4z & - & 2w & = & 9 \\ x & + & 3y & + & 9z & + & 7w & = & 24 \end{array}$$

Problem 5.
We consider a 4×5 homogeneous linear system $A\mathbf{x} = \mathbf{0}$, and $\operatorname{rk} A = 4$. How many solutions are there?

Exercise Problems

In addition to the Key Problems, you should try to do as many of the Exercise Problems as you can. They are usually problems from the textbook or exam problems. You can work with Exercise Problems and get help in the Exercise sessions/TA sessions on Mondays (Wednesdays). If you do not have time for all Exercise Problems in the following week, make sure to do them later in the semester as revision.

Problems from the textbook: [E] 1.1 - 1.17

Key Problems

Problem 1.
We consider the vectors $\mathbf{v}_1 = (1,3,4)$, $\mathbf{v}_2 = (-1,3,4)$, $\mathbf{v}_3 = (5,3,4)$, $\mathbf{v}_4 = (6,4,5)$, $\mathbf{v}_5 = (4,2,3)$.

a) Is $\mathbf{v}_3$ in $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$?

b) Express $\mathbf{v}_5$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ if possible.

c) Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ linearly independent vectors?

d) Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5$ linearly independent vectors?

e) Determine the dimension of $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5)$, and find a base of V.

f) Express $\mathbf{v}_5$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in a different way than in (b), if possible.

Problem 2.
Find a parametric description of the line through the points (1,2,1) and (4,5,3) in $\mathbb{R}^3$. Determine the intersection points (x,y,z) of this line and the plane $x - y + z = 6$.

Problem 3.
Determine $\dim V$ and $\dim W$ when $V = \text{Col}(A)$, $W = \text{Null}(A)$, and A is the 3×5 matrix A given below, and find a base of V and W:

$$A = \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 2 & 4 & -2 & -2 & -2 \\ 3 & 5 & -1 & -1 & -1 \end{pmatrix}$$

Problem 4.
Let A be a 8×8 matrix with rank given by $\text{rk}(A) = 7$ and let $\mathbf{b}$ be a vector in $\mathbb{R}^8$. Determine:

a) $\dim \text{Null}(A)$ and $\dim \text{Col}(A)$

b) The number of solutions of $A\mathbf{x} = \mathbf{0}$

c) The number of solutions of $A\mathbf{x} = \mathbf{b}$

d) The number of solutions of $A\mathbf{x} = \mathbf{0}$ that satisfies $x_1 + x_2 + \cdots + x_8 = 1$

Exercise problems

Problems from the textbook:	[E] 2.1 - 2.16
Exam problems:	[Midterm 10/2019] Question 1, 2, 8
	[Midterm 10/2022] Question 1, 2, 7

Key Problems

Problem 1.
Compute A^{-1} and A^2:

a) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}$

Problem 2.
Compute the determinant $|A|$:

a) $A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 0 & 1 & 3 & 0 \\ 4 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 \\ 0 & 3 & 1 & 0 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \\ 1 & q & q^2 \end{pmatrix}$

Problem 3.
Use minors to determine the rank of A, and find columns of A that form a base for the column space $\mathrm{Col}(A)$. You may combine minors with other methods if it is difficult to answer the questions using only minors.

a) $A = \begin{pmatrix} 4 & 1 & 1 & 3 & 7 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 & 0 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & 7 & 3 \\ 4 & 5 & 11 & 10 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 4 & -3 & 1 \\ 2 & 7 & 1 & 2 \\ 1 & 3 & 4 & 1 \end{pmatrix}$

Problem 4.
Use minors to find the rank of these matrices:

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ t & t & t \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 3 & 2 & -1 \\ s & 3 & 0 & 1 \\ 4 & 6 & 2 & 0 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & a & b \\ a & b & c \end{pmatrix}$

Exercise Problems

Problems from the textbook:	[E] 3.1 - 3.15
Exam problems:	[Midterm 01/2020] Question 1, 2, 4, 8
	[Final 11/2019] Question 1

Key Problems

Problem 1.
Find all eigenvalues of A, and a base for the eigenspace E_λ for each eigenvalue λ:

a) $A = \begin{pmatrix} 5 & 9 \\ 9 & 5 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & -4 \\ 3 & 0 \end{pmatrix}$

d) $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ e) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ f) $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Problem 2.
For the matrix A in Problem 1 a) - f), determine whether A is diagonalizable, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ when this is possible.

Problem 3.
Find the eigenvalues of A, and show that A is diagonalizable:

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4.
Use eigenvalues and eigenvectors of A to determine the limit of A^m when $m \to \infty$.

a) $A = \begin{pmatrix} 0.40 & 0.15 \\ 0.60 & 0.85 \end{pmatrix}$ b) $A = \begin{pmatrix} 0.77 & 0.46 \\ 0.23 & 0.54 \end{pmatrix}$

Problem 5.
Show that when A is a 3×3 matrix, then the characteristic equation of A can be written as $-\lambda^3+c_1\lambda^2-c_2\lambda+c_3 = 0$, where $c_1 = \text{tr}(A)$, $c_2 = M_{12,12} + M_{23,23} + M_{13,13}$ and $c_3 = \det(A)$. Hint: Write down the characteristic equation of a 3×3 matrix $A = (a_{ij})$ with general coefficients. Then use the formula to find the characteristic equation and the eigenvalues of the following matrices:

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 3 & 5 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ c) $A = \begin{pmatrix} 0 & 4 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise Problems

Problems from the textbook:	[E] 4.1 - 4.7
Exam problems:	[Midterm 10/2018] Question 1-6
	[Midterm 10/2022] Question 3,6,8

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